River Engineering

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1. Introduction

1.1 The nature of what we will and will not do – illuminated by some aphorisms and some people

“There is nothing so practical as a good theory” – stated in 1951 by Kurt Lewin (D-USA, 1890-1947): we want to solve practical problems, both in professional practice and research, and to do this it is helpful to have a theoretical understanding and a framework.

“The purpose of computing is insight, not numbers” – the motto of a 1973 book on numerical methods for practical use by the mathematician Richard Hamming (USA, 1915-1998). That statement has excited the opinions of many people. However, numbers *are* often important in engineering, whether for design, control, or other aspects of the practical world. A characteristic of many engineers, however, is that they are often blinded by the numbers, and do not seek the physical understanding that can be a valuable addition to the numbers. We might say simply: "The purpose of this course is insight into the behaviour of rivers; with that insight, numbers can be often be obtained more simply and reliably".

“It is EXACT, Jane” – a story told to the lecturer by a botanist colleague. She, non-numerical by training, had just seen the demonstration by an hydraulic engineer of a one-dimensional computational model of a river. She asked: “Just how accurate is your model?”. The engineer

replied intensely: "It is EXACT, Jane". That river was the most important one in Australia, the Murray River, $2\,375\,\mathrm{km}$ (Danube $2\,850\,\mathrm{km}$), maximum recorded flow $3\,950\,\mathrm{m^3s^{-1}}$ (Danube at Iron Gate Dam: $15\,400\,\mathrm{m^3s^{-1}}$). It has many tributaries, flow measurement in the system is approximate and intermittent, there is huge biological and fluvial diversity and irregularity. His model was NOT exact. In fact, nothing in these lectures will be exact. We are talking about the *modelling* of complex physical systems.

A further example of the sort of thinking that we would like to avoid: in the area of palaeo-hydraulics, some Australian researchers made a survey to obtain the heights of floods at individual trees. This showed that the palaeo-flood reached a maximum height on the River Murray at a certain position of $18.01\,\mathrm{m}$ (*sic*), Having measured the cross-section of the river, they applied the Gauckler-Manning-Strickler Equation to determine the discharge of the prehistoric flood, stated to be $7\,686\,\mathrm{m^3s^{-1}}$ ...

William of Ockham (England, c1288-c1348): Ockham’s razor is the principle that can be popularly stated as “when you have two competing theories that make similar predictions, the simpler one is the better”. The term razor refers to the act of shaving away unnecessary assumptions to get to the simplest explanation, attributed to 14th-century English logician and Franciscan friar, William of Ockham. We should make as few assumptions as possible, eliminating those that make no difference in the observable predictions of the hypothesis or theory, *still sufficiently answering the question*. That is, we should not *over*-simplify our approach.

The principle has inspired numerous expressions including “parsimony of postulates”, the “principle of simplicity”, the “KISS principle” (Keep It Simple, Stupid). Other common

restatements are:

Leonardo da Vinci (I, 1452–1519, world’s most famous hydraulician, also an artist): his variant short-circuits the need for sophistication by equating it to simplicity “Simplicity is the ultimate sophistication”.

Wolfang A. Mozart (A, 1756–1791): “Gewaltig viel Noten, lieber Mozart”, soll Kaiser Josef II. über die erste der großen Wiener Opern, die “Entführung”, gesagt haben, und Mozart antwortete: “Gerade so viel, Eure Majestät, als nötig ist.” (Emperor Joseph II said about the first of the great Vienna operas, “Die Entführung aus dem Serail”, “Far too many notes, dear Mozart”, to which Mozart replied “Your Majesty, there are just as many notes as are necessary”). The truthfulness of the story is questioned – Josef was more sophisticated than that ...

Albert Einstein (D-USA,1879-1955): “Make everything as simple as possible, but not simpler.” This is a better and shorter statement than Ockham!

Karl Popper (A-UK, 1902-1994): no number of experiments can confirm a scientific theory (Hume’s “Problem of Induction”), but a single counterexample can show the theory to be false. For example, consider the inference that “all swans we have seen are white, and therefore all swans are white”, before the discovery of black swans in Australia. We prefer simpler theories to more complex ones, as they apply to more cases than more complex ones, and are more easily *falsifiable*.

Thomas Kuhn (USA, 1922-1996): In *The Structure of Scientific Revolutions* found little evidence of scientists actually following a falsificationist methodology. He argued that scientists work in a series of paradigms, and that as science progresses, explanations tend to become more complex before a sudden *paradigm shift* offers radical simplification.

Paul Feyerabend (A-USA, 1924-1994): a student of Popper, ultimately rejected any prescriptive methodology, and argued that the only universal method characterising scientific progress was “anything goes!”

1.2 Summary

- We will use theory, but we will try to keep things simple, rather simpler than is often the case in this field, especially in numerical methods.
- Often our knowledge of physical quantities is limited, and approximation is justified.
- We will recognise that we are modelling.
- An approximate model can often reveal to us more about the problem.
- It might be thought that the lectures show a certain amount of inconsistency – in occasional places the lecturer will develop a more generalised and “accurate” model, paradoxically to emphasise that we are just modelling.
- We will attempt to obtain insight and understanding – and a sense of criticality.

1.3 Types of channel flow to be studied

An important part of this course will be the study of different types of channel flow.

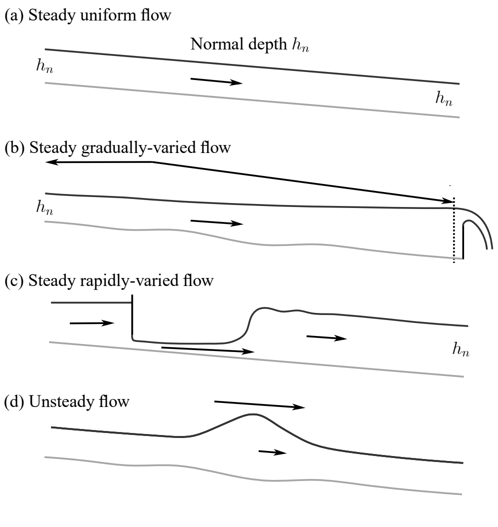


Figure 1.1: Different types of flow in an open channel

Case (a) – Steady uniform flow:

Steady flow is where there is no change with time, $\partial/\partial t \equiv 0$. Distant from control structures, gravity and resistance are in balance, and if the cross-section is constant, the flow is uniform, $\partial/\partial x \equiv 0$. This is the simplest model, and often is used as a first approximation for others.

Case (b) – Steady gradually-varied flow:

Where all inputs are steady but where channel properties may vary and/or a control may be introduced which imposes a water level at a certain point. The height of the surface varies along the channel. For this case we will study the governing differential equation that describes how conditions vary along the waterway, and we will obtain an approximate mathematical solution to solve general problems approximately.

Case (c) – Steady rapidly-varied flow:

Figure (c) shows three separate gradually-varied flow regions separated by two rapidly-varied regions: (1) flow under a sluice gate and (2) a hydraulic jump. The basic hydraulic approximation that variation is gradual breaks down in those regions. We can analyse them by considering energy or momentum conservation locally. In this course we will not be considering these – earlier courses at TUW have.

Case (d) – Unsteady flow:

Here conditions vary with time and position as a flood wave traverses the waterway. We will consider flood wave motion at some length.

1.4 Some possibly-surprising results

Effects of turbulence on dynamics

Where the fluid flow fluctuates in time, apparently randomly, about some mean condition, *e.g.* the flow of wind, water in pipes, water in a river. In practice we tend to work with mean flow properties, however in this course we will adopt empirical means of incorporating some of the effects of turbulence. Consider the x component of velocity at u a point written as a sum of the mean ($\bar{u}$) and fluctuating (u') components:

$$u = \bar{u} + u'.$$

By definition, the mean of the fluctuations, which we write as $\overline{u'}$, is

$$\overline{u'} = \frac{1}{T}\int_0^T u'\,dt = 0, \tag{1.1}$$

where T is some time period much longer than the fluctuations.

Now let us compute the mean value of the *square* of the velocity, such as we might find in computing the mean pressure on an object in the flow:

$$
\begin{aligned}
\overline{u^2} &= \overline{(\bar{u}+u')^2} = \overline{\bar{u}^2 + 2\bar{u}u' + u'^2}, \text{ expanding,}\\
&= \overline{\bar{u}^2} + \overline{2\bar{u}\,u'} + \overline{u'^2}, \text{ considering each term in turn,}\\
&= \bar{u}^2 + 2\bar{u}\,\overline{u'} + \overline{u'^2}, \text{ but, as } \overline{u'} = 0 \text{ from equation (1.1),}\\
&= \bar{u}^2 + \overline{u'^2}.
\end{aligned}
\qquad (1.2)
$$

hence we see that the mean of the square of the fluctuating velocity is not equal to the square of the mean of the fluctuating velocity, but that there is also a component $\overline{u'^2}$, the mean of the fluctuating components. We will need to incorporate this.

Pressure in open channel flow – effects of resistance on flows over steep slopes

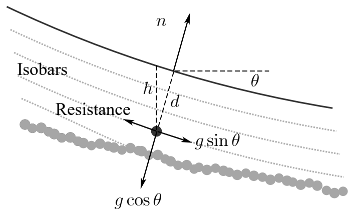


Figure 1.2: Channel flow showing isobars and forces per unit mass on a fluid particle

Consider Figure 1.2 showing an open channel flow with forces per unit mass acting on a particle. The figure is drawn, showing that in general, the depth is not constant, and the bed is not parallel to the free surface. The surface is an isobar, a line of constant pressure, $p = 0$. In the flow, other isobars will approximately be parallel to this, while the channel bed is not necessarily an isobar. We consider the vector Euler equation

for the motion of a fluid particle

$$\text{Acceleration} = -\frac{1}{\rho} \times \text{Pressure gradient} + \text{Body forces per unit mass}\ ,$$

where ρ is the fluid density. In a direction parallel to the free surface, the pressure is constant and there is no pressure gradient. The acceleration of the particle will be given by the difference between the component of gravity $g\sin\theta$ and the resistance force per unit mass. We usually do not know the details of that, so there is little that we can say. Now considering a direction perpendicular to that, given by the co-ordinate n on the figure, there is very little acceleration, so we assume it to be zero, and so we obtain the result

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial n} - g\cos\theta.$$

Now integrating this with respect to n between a general point, such as $n = -d$ at the particle shown, and $n = 0$ on the surface where $p = 0$ we obtain

$$p = \rho g \cos\theta \times d.$$

It is much more convenient to measure all elevations vertically, and so we use h, such that $d = h\cos\theta$, and we obtain the general expression for pressure

$$p = \rho g h \cos^2\theta. \qquad (1.3)$$

The general result of equation (1.3) for flow on a finite slope seems to have been forgotten by many. In general, pressure in flowing water is not "hydrostatic". However in this course, bed slopes are small enough that we will use it.

Hydrostatic approximation

An almost-universal assumption in river engineering is that the slope of the surface is small enough such that $\cos^2\theta \approx 1$, and we can use the *hydrostatic approximation*, where the surface slope is so small, as if it were obtained from a static fluid where the surface is horizontal,

$$p = \rho g h = \rho g\,(\eta - z)\,, \qquad (1.4)$$

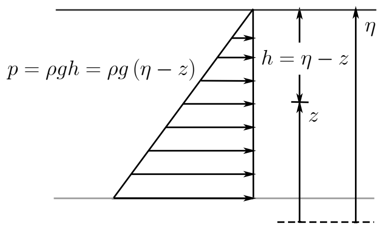


Figure 1.3: Hydrostatic pressure distribution

where z is the vertical co-ordinate and η is its value at the free surface. Such a pressure distribution is called *hydrostatic*, equivalent to that of water which is not moving, such that pressure p at a point is given by the height of water above, $p = \rho g h$, where ρ is fluid density ($\approx 1000\,\mathrm{kg\,m^{-3}}$ for fresh water), $g \approx 9.8\,\mathrm{m s^{-2}}$ is gravitational acceleration, and h is the vertical height of the surface above the point. We have seen that this is not necessarily the case in flowing water for cases such as spillways or block ramps, which are steep.

2. Resistance in river and other open channel flows

The resistance to the flow of a stream is a very important quantity in river mechanics – and is almost always poorly known.

2.1 The channel flow formula

We consider a simple theory based on force balance and some classical fluid mechanics experiments to obtain a flow formula for a wide rectangular channel.

Here, the fundamental flow formula for steady uniform flow in channels is developed from theory and experimental results. We show that the traditional flow formulae of Gauckler-Manning and Chézy-Weisbach are simply different approximations to that.

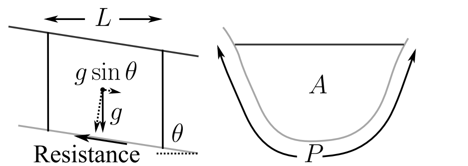


Figure 2.1: Uniform flow in a channel, showing resistance and gravity forces on a finite length, plus cross-section quantities

Consider a horizontal length L of uniform channel flow, inclined at a small angle θ to the horizontal, with cross-sectional area A. The volume of the element is AL, the vertical gravitational force on the water is $\rho g A L$, where ρ is fluid density and g is gravitational acceleration. The component of this along the slope is $\rho g A L \sin\theta$. The resistance force along the slope, of length $L/\cos\theta$ is $\tau P L/\cos\theta$, where τ is the mean resistance shear stress, assumed uniformly distributed around the wetted perimeter P around which it acts. Equating gravitational and resistance components gives $\tau P L/\cos\theta = \rho g A L \sin\theta$. To high accuracy for small θ,

$\cos\theta \approx 1$ and $\sin\theta \approx \tan\theta = S$, the slope, giving

$$\frac{\tau}{\rho} = g\frac{A}{P}S. \tag{2.1}$$

Our problem is now to express shear stress τ in terms of flow quantities.

One of the most famous series of experiments in fluid mechanics was performed by Johann Nikuradse at Göttingen in the 1930s, who studied the flow of fluid over uniformly-rough sand grains. The fluid was actually air, and the sand grains were actually in circular pipes, but the results are still valid enough.

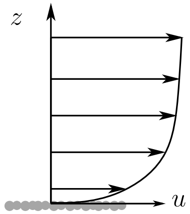


Figure 2.2: Idealised logarithmic velocity profile in turbulent flow over rough bed

With those results, for a wide channel of depth h with sand grains of size k_{s}, the velocity distribution for fully rough flow (no effects of viscosity), the *universal velocity distribution* can be written:

$$u = \frac{u_*}{\kappa}\ln\frac{30\,z}{k_{\mathrm{s}}}, \tag{2.2}$$

in terms of the shear velocity $u_* = \sqrt{\tau/\rho}$, the von Kármán constant $\kappa \approx 0.4$, the vertical co-ordinate z, and where the factor of 30 is for closely-packed uniform sand grains. It varies with other types of boundary roughness. The mean velocity U is obtained by integrating between 0 and h, such that

$$U = \frac{1}{h}\int_0^h u\,dz = \frac{u_*}{\kappa}\ln\frac{30/\mathrm{e}}{k_{\mathrm{s}}/h}. \tag{2.3}$$

where $\mathrm{e} = \exp(1) = 2.718\ldots$ is Euler's number. The result has been obtained in terms of *relative roughness* k_{s}/h. We replace $u_* = \sqrt{\tau/\rho}$ using equation (2.1) to give

$$U = \frac{1}{\kappa}\sqrt{g\frac{A}{P}S}\left(\ln\frac{30/\mathrm{e}}{k_{\mathrm{s}}/h}\right). \tag{2.4}$$

We have obtained something very useful – a formula for the mean flow velocity in a wide rectangular channel of depth h, slope S, and relative roughness k_{s}/h. We have used simple mechanics plus an empirical laboratory result. Surprisingly, the formula is explicit in terms of physical quantities – we have not had to assume a value like the Manning coefficient n or Strickler coefficient $k_{\mathrm{St}} = 1/n$!

2.2 Channels of arbitrary section

To obtain the equivalent formula for channels of any section we consider velocity distributions in real streams and develop an approximation giving a general flow formula.

The previous section was for a wide channel with an idealised logarithmic velocity distribution. In nature, for channels of any general cross-section there is the problem that the velocity has a maximum somewhere below the surface, and in general the isovels are something like Figure 2.3. Even in straight channels there are longitudinal vortices such that in the centre of the channel the maximum in velocity, which would be expected to be at the surface, is actually at a lower position.

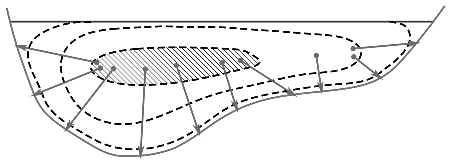


Figure 2.3: Cross-section of flow showing isovels and, for a number of points on the bed, where the fastest-likely fluid comes from and how far it travels, the effective length scale for resistance calculations.

To obtain a flow formula for channels of any cross-section, we hypothesise that the effective depth h for resistance calculations is the typical distance from points with the highest velocity to the nearest point on the bed, as suggested by the arrows on the figure. The fluid flow on the boundary, where resistance occurs, would be similar to that in a channel, not of the actual mean depth, but the mean length of the arrows.Typical length scales as shown by the arrows are somewhat *smaller* than the overall mean depth of flow. Our problem is then, how to approximate that distance? We examine the approach suggested by the lecturer (Fenton 2011).

We consider the experimental data for the vertical position of the locus of velocity maxima in *rectangular* channels from Yang, Tan & Lim (2004). From their results we can obtain a formula for the mean elevation of the velocity maximum $z_{\max}/h$, as a function of aspect ratio (channel width B divided by depth h).

There is another length scale in equation (2.4) which is the ratio of area to perimeter A/P, which, as $P > B$, should be smaller than the mean depth A/B, so it might be a candidate for the depth scale as experienced by the bed. We calculate the ratio:

$$\frac{A/P}{h} = \frac{Bh}{(B+2h)h} = \frac{B/h}{B/h+2}. \tag{2.5}$$

Both the expression for $(A/P)/h$ and the experimental formula for $z_{\max}/h$ are plotted in Figure 2.4. *Remarkably*, the two coincide closely over a wide range of aspect ratios, so that we have found the quantity A/P mimics the behaviour of $z_{\max}$, which we have suggested is, instead of h, the apparent depth that the flow on the bed experiences. This suggests that in equation (2.4), instead of h in the term from the velocity distribution, we can use A/P. We cannot claim that this is a justification as strong as it looks, but we have seen that A/P already appears in the equation, appearing naturally in the simple mechanical equilibrium calculation.

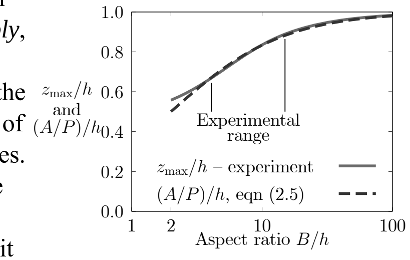


Figure 2.4: Rectangular channels: dimensionless mean elevation of $z_{\max}/h$ and the effective depth $(A/P)/h$

For channels that are not rectangular we have presented no results. Our suggestion is that A/P will still be a plausible approximation, and it already appears in equation (2.4). The use of A/P was justified by Keulegan (1938), however while that work mathematically correctly integrated logarithmic velocity distributions perpendicular to parts of various shapes of cross-section, it did not give any attention to the real velocity distributions, especially ignoring the phenomenon of the velocity maximum being below the surface.

We have shown that A/P is approximately equal to the mean distance of the maximum velocity from the bed, so that the flow on the bed is similar to that of a channel of depth A/P. In channels that are wide, which is most, $P \approx B$ and A/P is about the same as the geometric mean depth A/B. Our suggested channel flow formula, replacing h by A/P in equation (2.4) is

$$U = \frac{Q}{A} = \frac{1}{\kappa}\sqrt{g\frac{A}{P}S}\left(\ln\frac{30/\mathrm{e}}{k_\mathrm{s}/\left(A/P\right)}\right). \tag{2.6}$$

If we knew an accurate value of k_s, this is probably the formula that we should use, as it is in terms of physical quantities that we know or we can approximate, including the equivalent grain size k_s. For example, if we were studying the Danube, we might simply use the typical grain size $k_\mathrm{s} = D = 0.02\,\mathrm{m}$. Traditional practice, however is often to use Chézy-Weisbach and Gauckler-Manning-Strickler formulae, which introduce resistance coefficients which, while more general, and allow for other forms of resistance such as vegetation and bed forms, are more empirical and their physical significance less clear.

2.3 The Chézy-Weisbach flow formula

The oldest flow formula is that of Chézy . Here it is written in terms of g and the Weisbach dimensionless resistance coefficient λ.

We write shear stress τ in terms of the result obtained from the Darcy-Weisbach formulation of flow resistance in pipes,

$$\frac{\tau}{\rho} = \tfrac{1}{8}\lambda U^2, \tag{2.7}$$

where λ is the Weisbach dimensionless resistance coefficient, expressing the relationship between velocity and stress. The factor of $1/8$ is necessary to agree with the Darcy-Weisbach energy formulation of pipe flow theory in *circular* pipes where $A/P = \text{Diameter}/4$. From our simple force balance we already have equation (2.1): $\tau/\rho = \sqrt{g\left(A/P\right)S}$. Eliminating τ/ρ between equation (2.7) and this gives the Chézy-Weisbach flow formula

$$U = \frac{Q}{A} = \sqrt{\frac{8g}{\lambda}\frac{A}{P}S} = C\sqrt{\frac{A}{P}S}, \tag{2.8}$$

where we have also written it in terms of C, the Chézy coefficient, giving Chézy's flow formula, named after the French military engineer who first presented it in 1775. We see that $C = \sqrt{8g/\lambda}$.

Comparing our flow formula (2.6) with (2.8) shows that what we have done is to obtain a formula

for the dimensionless coefficient λ:

$$\lambda = \frac{8\kappa^2}{\left(\ln\frac{30/\mathrm{e}}{k_\mathrm{s}/(A/P)}\right)^2} \approx \frac{8\kappa^2}{\ln^2\left(11/\varepsilon\right)}, \tag{2.9}$$

where we have introduced the symbol ε for the relative roughness

$$\varepsilon = \frac{k_\mathrm{s}}{A/P} = \frac{\text{Equivalent grain size}}{\text{Hydraulic mean depth}} \approx \frac{\text{Grain size}}{\text{Depth}}. \tag{2.10}$$

We now have a flow formula for steady uniform flow in a channel based on simple theory, experimental observations, and a bold approximation (A/P instead of depth).

Relative unimportance of grain size

In fact, λ, although all-important for us, is relatively slowly varying with grain size. Consider a small change in the relative roughness $\varepsilon + \delta\varepsilon$. The relative change $\delta\lambda$ is

$$\frac{\delta\lambda}{\lambda} = \frac{\left(\ln\frac{11}{\varepsilon}\right)^2}{\left(\ln\frac{11}{\varepsilon+\delta}\right)^2} - 1 \approx \frac{2}{\ln\frac{11}{\varepsilon}}\frac{\delta\varepsilon}{\varepsilon}$$

having expanded the logarithm as a power series $\ln\left(\varepsilon + \delta\varepsilon\right) \approx \ln\varepsilon + \delta\varepsilon/\varepsilon$ and performed some elementary series operations. Now for a value of $\varepsilon = 0.001$ (a $1\,\mathrm{mm}$ grain in $1\,\mathrm{m}$ of water), a relative change of $\delta\lambda/\lambda$ is only 20% of the relative size change $\delta\varepsilon/\varepsilon$. Even for a much rougher case of $\varepsilon = 0.1$, $\delta\lambda/\lambda$ is only 40% of the relative size change $\delta\varepsilon/\varepsilon$. It does not matter so much if we cannot specify the bed conditions so very accurately.

An approximation to our formula

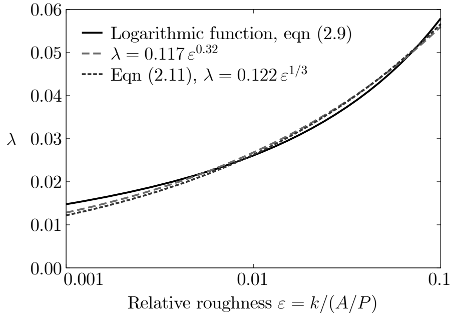


Figure 2.5:

On Figure 2.5 is shown how the resistance coefficient λ varies as a function of relative roughness ε, given by equation (2.9) from experimental fluid mechanics. It is actually possible to approximate that curve closely using a monomial function $\lambda = a\,\varepsilon^\mu$. The best values of a and μ can be found by performing a least-squares fit. Using 11 points equally-spaced in $\log\varepsilon$ between $\varepsilon = 0.001$ and 0.1, the result obtained is $\mu = 0.32$ and $a \approx 0.117$, the approximation plotted on Figure 2.5, showing that it is quite accurate. The value of μ obtained is so close to $1/3$ that for convenience the logarithmic function is now approximated again, this time by the function $b\,\varepsilon^{1/3}$, where b is a constant which can also be determined by least-squares, giving $b \approx 0.122$ such that we write

$$\lambda = 0.122\,\varepsilon^{1/3} = 0.122\left(\frac{k_\mathrm{s}}{A/P}\right)^{1/3}, \tag{2.11}$$

plotted on Figure 2.5, showing that this is also quite a good approximation to the logarithmic function, considering the uncertainty that one has in practice in knowing $\varepsilon = k_\mathrm{s}/\left(A/P\right)$.

2.4 The Gauckler-Manning-Strickler formula

We consider the approximation to the formula we obtained theoretically and find that we have obtained the Gauckler-Manning-Strickler formula, including a theoretical prediction of Strickler's formula for the effect of boundary grain size.

If we substitute the approximation (2.11) into the theoretical flow formula, equation (2.8) and re-write, we obtain

$$U=\frac{Q}{A}=\frac{8.1\sqrt{g}}{k_{\mathrm{s}}^{1/6}}\left(\frac{A}{P}\right)^{2/3}\sqrt{S}, \tag{2.12}$$

we obtain the most widely-used resistance formula in river engineering, **the Gauckler-Manning (GM) formula**, (previously just "Manning"), written using traditional notation as

$$U=\frac{Q}{A}=\frac{1}{n}\left(\frac{A}{P}\right)^{2/3}\sqrt{S}=k_{\mathrm{St}}\left(\frac{A}{P}\right)^{2/3}\sqrt{S}, \tag{2.13}$$

where n is the Manning coefficient and $k_{\mathrm{St}}=1/n$ is the Strickler coefficient, used in German-speaking countries. The formula was originally suggested by Gauckler, and then by Manning, in the nineteenth century, having observed that variation with A/P seemed to be better represented by $(A/P)^{1/3}$, than the $(A/P)^{1/2}$ of the Chézy-Weisbach formula (2.8) with constant coefficients.

In equation (2.12) we have, in addition, obtained an explicit expression for the Strickler/Manning

coefficient, obtained from theory and experiments of the early twentieth century:

$$k_{\mathrm{St}}=\frac{1}{n}=\frac{8.1\sqrt{g}}{k_{\mathrm{s}}^{1/6}}. \tag{2.14}$$

A similar result was obtained by Strickler (a century ago, without optimising software, and *before* the fluid mechanics advances we referred to), based entirely on hydraulic experiments. Instead of the equivalent sand grain roughness k_{s} that Nikuradse and we have used, he considered equivalent mean diameter D, from $D=0.1\,\mathrm{mm}$ to $D=300\,\mathrm{mm}$, (where that diameter was sometimes calculated from alluvial gravel with relative lengths of the three axes 1:2:3). For the numerical coefficient (8.1 in eqn 2.14) he obtained a value of $4.75\sqrt{2}\approx 6.7$, giving his expression

$$k_{\mathrm{St}}=\frac{1}{n}=\frac{6.7\sqrt{g}}{D^{1/6}}. \tag{2.15}$$

The expression (2.14) here has been obtained by a quite different route, and the agreement between the two expressions, one based on sand grains glued to the inside of a circular pipe carrying air, is encouraging. Of course, for river engineering purposes, Strickler's result (2.15) is to be preferred.

We call the GM formula, equation (2.13) written with the expression (2.15) for the resistance coefficient,the **Gauckler-Manning-Strickler (GMS) formula**.

Sensitivity to boundary particle size

As earlier, we examine the effect of uncertainty or variability in the size of the boundary particles (and any perceived ambiguity between k_{s} and D), replacing D by $D+\delta D$ and using a power series expansion of equation (2.15)

$$\frac{\delta n}{n}=\left(1+\frac{\delta D}{D}\right)^{1/6}-1=\frac{1}{6}\frac{\delta D}{D}+\ldots,$$

and so a fractional change in boundary particle size gives a relative change of $1/6$ in resistance. Again, resistance varies slowly with grain size. Precise knowledge of that is not so important.

Revision As we have space, this is a good place to revise some elementary power series results. For ε small:

Binomial theorem:

$$(1+\varepsilon)^n=1+n\varepsilon+O\left(\varepsilon^2\right).$$

Series for logarithm function

$$\ln(1+\varepsilon)=\varepsilon+O\left(\varepsilon^2\right).$$

Series for exponential function:

$$\exp(\varepsilon)=\mathrm{e}^{\varepsilon}=1+\varepsilon+O\left(\varepsilon^2\right).$$

We have used the "Big O" notation to show the size of neglected terms.

Test of logarithmic and GMS formulae

Comparison with a series of experiments validates the Strickler approach, giving an explicit flow formula for a variety of channel boundaries.

To test the accuracy of the GMS formula compared with the logarithmic formula we obtained from experiment, equation (2.12), we consider the results of Strickler (1923, Beilage 4). Strickler considered results from nine very different channels. For each the lecturer calculated the equivalent k_{s} or D, constant for each channel, by least-squares fitting of the appropriate flow formula to the points, with results shown in the figure. The Gauckler-Manning-Strickler formula gives agreement generally as good as our logarithmic formula obtained from fluid mechanics experiments.

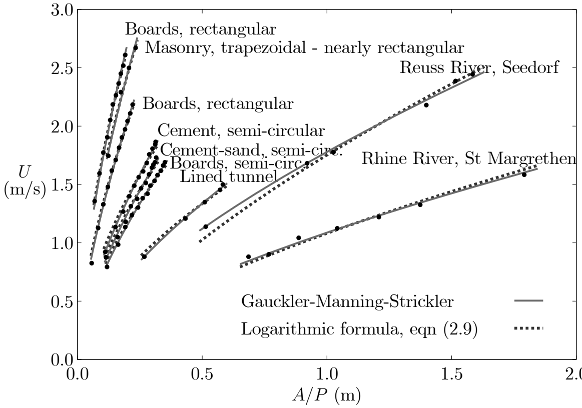


Figure 2.6: Strickler's results approximated by two flow formulae

2.5 Using the Weisbach notation

It is better to use a simpler formulation of the flow formula in which forces and the mechanics are clearer: the Weisbach form of the flow formula (2.8) has several advantages.

We have established the general validity of the GMS formula for river beds composed of regular soil particles or artificial boundaries. However

- The GM formula involves coefficients that are of an ugly non-scientific form – we do not know what k_{St} or n really are. They cannot be combined rationally in more complicated situations.
- Also, they have the significant problem that they have difficult units (n: $\mathrm{L}^{-1/3}\mathrm{T}$). This problem is a difficult one in the three countries (Liberia, Myanmar, and USA) which use the confusing Imperial units, leading to special formulae.

Here we prefer to write the channel flow formula in the Weisbach form, equation (2.8), but introducing the coefficient

$$\Lambda = \frac{\lambda}{8} \tag{2.16}$$

to eliminate the annoying occurrences of the factor of 8 throughout the equations which are due to the original introduction of λ for head loss in circular pipes (area $\propto \pi/4$, head $\propto \frac{1}{2}u^2$). The significance of Λ follows from equation (2.7):

$$\frac{\tau}{\rho} = \tfrac{1}{8}\lambda U^2 = \Lambda U^2. \tag{2.17}$$

Our form of the Weisbach formula is

$$U = \frac{Q}{A} = \sqrt{\frac{g}{\Lambda}\frac{A}{P}S}, \tag{2.18}$$

where the resistance coefficient Λ is dimensionless and has a clear physical significance relating shear stress and mean velocity, equation (2.17). If we need to calculate a value from the GM form we obtain

$$\Lambda = \frac{gn^2}{(A/P)^{1/3}}. \tag{2.19}$$

Or, if we want to use the Strickler formula for resistance we substitute equation (2.15) for $n|k_{\mathrm{St}}$ to obtain the simple dimensionless expression

$$\Lambda = 0.0223\left(\frac{D}{A/P}\right)^{1/3}, \tag{2.20}$$

where the coefficient might be more reasonably written as 0.022.

The GMS equation evaluated in the Weisbach form (2.18), possibly calculating a dimensionless resistance coefficient from either equation (2.19) or (2.20), particularly in the latter form, has advantages:

- If we were uncertain about the magnitude of the resistance coefficient Λ – and we usually are – it provides some physical significance. The fact that it is based on a relationship between stress (force) and velocity squared means that we can rationally introduce and combine other forms of resistance such as vegetation and structures.

A simple example is that of a laboratory flume (channel) with a sand bed and glass sides. From previous experimental work we would have values of λ for sand and for glass and could combine them rationally, as forces can be added. There is no rational theory for compound values of n and k_{St}. In fact there are very bad abuses for that – to be described later.

- The presence of bed forms such as ripples, dunes, *etc*, cannot be so simply calculated, but here one could also linearly separate and/or combine contributions in terms of different Λ.
- Expressing Λ in terms of $D/(A/P)$ is a good basis: we could experiment with estimated values, examining the effect of different grain size assumptions on the calculated flow.
- To use the formula, we do *not* have to engage in any of the following traditional methods of calculating resistance coefficient n or k_{St}:
  - Think of a value, which is difficult, as it has no simple physical significance.
  - Look at pictures of rivers in standard references such as Chow (1959, §5-9&10). We do not see the underwater conditions determining the resistance.
  - Ring a friend to see what they used for a similar stream $20\,\mathrm{km}$ distant some years ago. The lecturer was on an Australian Committee for Stream Resistance. We set up an information centre where we hoped to collect data on n from all over the country. We only ever received requests for data going in the other direction: "What do you think n is for river X between A and B?". We had almost no information to distribute!

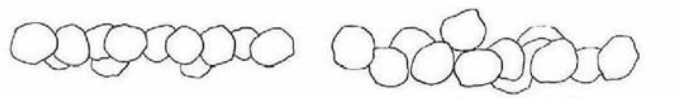
2.6 General situations

The problem of the current momentary resistance in the stream is actually a very difficult and uncertain one, with large variation. We plot a diagram with a large number of field studies showing the variation and how one might use the figure and empirical formulae to obtain a resistance coefficient.

Causes of resistance

In many cases the conditions in the river are more complicated than just a layer of uniform regular particles. For example:

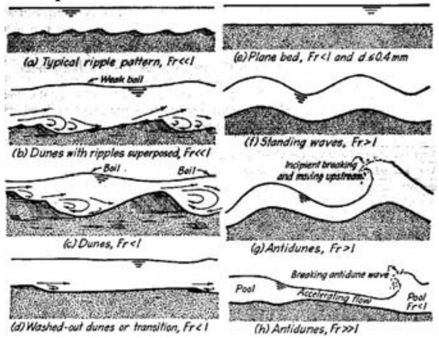
- Irregular and variable nature of the bed particle arrangement.



Coplanar "armoured" bed - low resistance   Random protruding grains - higher resistance

The variability of resistance in real streams is often much greater than has been realised, for the arrangement of the bed "grains" or "particles" (even if they are $30\,\mathrm{cm}$ boulders) is very important, and can change continuously, depending on the flow history. Nikuradse's experiments were for sand grains levelled so that their tops were co-planar, and hence most of the particles were shielded from the flow and resistance was small. That is what a bed looks like after a long period of constant flow, when any individually-projecting grains have been removed, because the force on them was larger. The bed is said to have been "armoured" – not only is the resistance small, but individual grains are hard to remove. After a sudden increase of flow, particles are more likely to have been dislodged, moved, and deposited, leaving a random surface, where those most projecting exert larger force on the fluid and resistance is greater.

- Bed forms – ripples, dunes, anti-dunes *etc.*
- The bed-forms which can develop if the bed is mobile will also contribute to variable resistance.



Typical bedforms (after Richardson and Simons)

- Particle movement – if the grains are actually moving, then the force required to move the grains

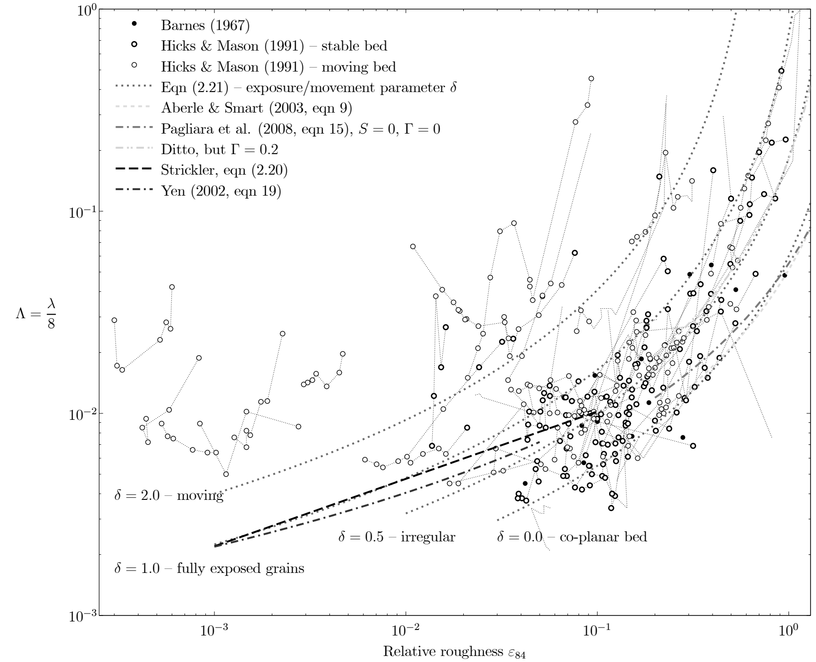
appears to the water as an additional stress, whether they are moving along the bed, rolling, jumping, or carried suspended in the flow.

- Vegetation – trees (standing and/or fallen), grasses, reeds *etc*
- Meandering

Results for resistance coefficients in real rivers

Here we attempt to obtain understanding and a formula for the resistance coefficient using results from a number of field measurements. To compare with several experimental works, we use the Chézy-Weisbach formulation. We considered the results of Hicks & Mason (1991), a catalogue of 558 stream-gaugings from 78 river and canal reaches in New Zealand, of which 55 were sites with grading curves for boundary material, so that particle sizes were known. Neither vegetation nor bed-form resistance can be isolated. Hicks & Mason based their approach on Barnes (1967), who provided values of Manning's resistance coefficient $n = 1/k_{\mathrm{St}}$ for a single flow at each of 50 separate river sites in the United States of America, of which boundary material details were given for 14. We also include those results here.

From both catalogues we took the values of D_{84}, the boundary particle size for which 84% of the material was finer, and from the values of A/P, calculated the relative roughness $\varepsilon_{84} = D_{84}/(A/P)$, and used the measured values of Chézy's C to calculate values of λ ($C = \sqrt{8g/\lambda}$). We have plotted them for the parameter $\Lambda = \lambda/8$.



- Many of the results from each study are for large bed material $\varepsilon_{84} > 0.1$, possibly a reflection of the hilly and mountainous nature of New Zealand and Pacific North-West of the United States of America (and which applies to Austria ...).
- There is a wide scatter of results. But not all that very wide if we consider that the streams range from large slow-moving rivers with extremely small grains to mountain torrents with $30\,\mathrm{cm}$ boulders. Most of the results, unless the grains are moving, fall between $\Lambda \approx 0.005$ and 0.02.
- There is, as we have seen, slow variation with relative roughness: an increase in ε by a factor of 10 leads to an increase in Λ of about 2. We have already shown this theoretically.
- The vertical scatter of the points, we believe, show the effects of bed arrangement and particle movement. We have plotted four curves using an arbitrary parameter δ. They have been drawn using the expression, found by trial and error:

$$\Lambda = \frac{0.06 + 0.06\,\delta}{(1.0 - 0.6\,\delta - \ln \varepsilon_{84})^2}, \tag{2.21}$$

  with values of $\delta = 0$, 0.5, 1, and 2, which we believe identifies the state of the stream bed. This will now be explained.
- We hypothesise that the experimental points showing the lowest resistance are those with beds where the tops of the particles are relatively co-planar such that the bed is *armoured*. We assigned $\delta = 0$ to this state, and used that in equation (2.21) to plot the curve. This is supported by the fact that the for large ε_{84} the curve passes into two sets of experimental curves for co-planar beds, Aberle & Smart (2003) and Pagliara *et al.* (2008) for their parameter $\Gamma = 0$.

- The second curve from the bottom, $\delta = 0.5$ substantially coincides, for large ε_{84}, with a curve corresponding to exposed boulders on top of the bed occupying 0.2 of the surface area obtained by Pagliara *et al*. (2008). This is an intermediate state, with a greater number of these grains thus exposed, the resistance is greater.
- Substituting $\delta = 1$ in equation (2.21) gives the third curve on the figure, approximately bounding above what we believe is an identifiable grouping of particles. This is probably the state for the maximum resistance for a stable bed corresponding to a maximum state of disorder, with exposed grains occupying something like 50% of the surface area. Any more such grains will cause shielding of particles, the bed will start to resemble the co-planar case, the resistance will actually be reduced, and with a lower δ.
- For points above the third curve almost all experimental points had shear stresses greater than the critical one necessary for movement, so that the bed is moving. If particles move, not only do many particles protrude above others, increasing the stress, but there is the additional force required to maintain the sliding and rolling and jostling of all the particles. Hence, the resistance is greater. And, if there is a need to maintain particles in suspension, that will contribute also to resistance, which is more likely for smaller particles as the experimental points show. We have shown the fourth curve, notionally above which the bed moves, for $\delta = 2$.
- Further evidence supporting our assertions is obtained from two supposedly general formulae:
  - Strickler's formula, equation (2.20), $\Lambda = 0.0223\,\left(D/\left(A/P\right)\right)^{1/3}$, which is plotted with D_{84}, giving a straight line of gradient $1/3$ on the logarithmic axes.
  - The approximation proposed by Yen (2002, eqn 19) for λ, who considered results from

a number of experimental studies using fixed impermeable beds. We used his formula, converted to $\Lambda = \lambda/8$, used an infinite Reynolds number, and converted his equivalent sand roughness $\varepsilon_s = 2\varepsilon_{84}$ (a not-unreasonable value – there is much debate ...).

Both these general curves start (left-to-right) from our curve $\delta = 1$, for small particles which will have the maximum state of randomness, as for such particles the tops cannot be levelled, to the second curve $\delta = 0.5$ for larger particles ($\varepsilon_{84} \approx 0.1$), more likely to be levelled in the laboratory experiments.

It is interesting that Strickler's formula, obtained for a variety of conditions as shown so convincingly in Figure 2.6 and which has been widely recommended, as we did above, seems to contribute relatively little for the variety of streams and conditions plotted on page 31.

Hopefully the figure and approximating curves have given us an idea of the magnitudes and variation of Λ, and maybe even some results for use in practice. It has certainly given us an idea of the magnitude of the problem of predicting the resistance coefficient!

2.7 Two complications – unsteady non-uniform flows and compound channel sections

We consider more complicated situations. The important problem of a compound cross-section is difficult to solve rationally. Existing simple methods have been abuses of science.

Non-uniform and unsteady flows

We will be considering flows which are not uniform (vary with position x) and those which are neither uniform nor steady (vary also with time t). As the length scale of river flows is much longer in space than the cross-sectional dimensions and the time scale of disturbances is much longer than that of local turbulence, we will assume that the boundary stress at each place and at each time is given by the local immediate flow conditions of velocity, in terms of discharge Q and area A. From equation (2.17) we have

$$\frac{\tau}{\rho} = \Lambda U^2 = \Lambda \left(\frac{Q\,(x,t)}{A\,(x,t)}\right)^2. \tag{2.22}$$

Compound cross-sections

We consider compound cross-sections such as shown in Figure 2.7. Using equation (2.22) we can obtain an expression for the boundary shear *force* per unit length of channel in each component. Multiplying by the perimeter of each part:

$$\tau_i P_i = \rho \Lambda_i \frac{Q_i^2 P_i}{A_i^2}, \quad \text{for} \quad i = 1, 2, \ldots$$

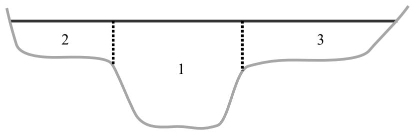


Figure 2.7: Compound cross-section

Note that the internal shear forces across faces 1-2 and 1-3 cancel. However, we do not know the individual discharges Q_i of the three sections. An approximation would be to neglect interfacial

shear and obtain the discharges for each component from the GMS equation.

The total gravitational component, force per unit length is

$$\rho g S \sum\nolimits_{i=1}^{3} A_i = \rho g S A,$$

where we have assumed that the slope S is the same for each part and where A is the total area.

To solve the steady flow problem then, if we write $Q_i = \theta_i Q$, where Q is the total flow, summing the components due to boundary force and equating to the gravitational component we obtain

$$Q^2 \sum\nolimits_{i=1}^{3} \frac{\Lambda_i P_i \theta_i^2}{A_i^2} = g S A,$$

which we could use to calculate the total flow Q or make more complicated deductions. There is an important problem: we do not know what the θ_i are.

Warning: do not use the apparently simple formulae for combining k_{St} or Manning's n appearing in many books, as exemplified by the list of 17 different compound or composite section formulae in Yen (2002, table 3), or Cowan's formula (see page 36 of Yen's paper) $n = \left(\sum_i n_i\right) m$, where the n_i are different contributions from surface roughness, shape and size of channel cross-section, *etc.*. That is irrational nonsense. The proper way to proceed is by a linear sum of the forces as we have done here, however we have seen that this has problems.

2.8 Computation of normal flow

The common practical problem of calculating the water depth for a given flow rate is considered. A computational method is developed and applied.

At last we turn to a common practical problem in River Engineering. "Normal flow" is the name given to a uniform flow, and the depth is called the normal depth. If the discharge Q, slope S, resistance coefficient $k_{\mathrm{St}} = 1/n$, and the relationship between area and depth and perimeter and depth are known, the GMS formula becomes a transcendental equation for the normal depth h. The problem is to solve the equation for h.

A numerical method

Any method for the numerical solution of transcendental equations can be used, such as Newton's method. Here we develop a simple method based on *Direct Iteration*, where we develop a trick, giving us rapid convergence. It is simpler to present this method using the classical GM formulation of the problem which the lecturer so criticised – it brings all variation with A/P together so that a term can be isolated.

Consider the GM formula in the conventional form, written now

$$Q = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}\sqrt{S}\,.$$

We divide both sides by $h^{5/3}$, and showing functional dependence of A and P on h:

$$\frac{Q}{h^{5/3}} = \frac{1}{n}\sqrt{S}\frac{(A(h)/h)^{5/3}}{P^{2/3}(h)}.$$

The term $A(h)/h$ is approximately the width of the channel. For wide channels (*i.e.* rather wider than they are deep, a common case) it varies little with h, and neither does the perimeter $P(h)$. So, the right side of the equation varies slowly with h. Now, by isolating the $h^{5/3}$ term and then taking the $3/5$ power of both sides of the equation, we obtain the equation in a form suitable for direct iteration

$$h = \left(\frac{nQ}{\sqrt{S}}\right)^{3/5} \times \frac{P^{2/5}(h)}{A(h)/h}, \tag{2.23}$$

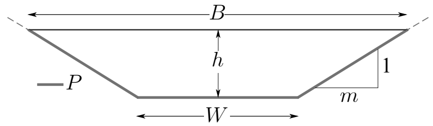
where the first term on the right is a constant for any particular problem, and the second term varies slowly with depth – a primary requirement that the direct iteration scheme be convergent and indeed be quickly convergent. We could use Strickler's formula (2.15) for n in terms of D.

For an initial estimate we suggest making a rough estimate of the approximate width B_0 and so, making a wide channel approximation, setting $A(h)/h \approx B_0$ and $P(h) \approx B_0$ in the general scheme of (2.23) gives

$$h_0 = \left(\frac{nQ}{B_0\sqrt{S}}\right)^{3/5}. \tag{2.24}$$

Experience with typical trapezoidal sections shows that the method works well and is quickly convergent. Again, we could use Strickler's formula (2.15) for n in terms of D.

Trapezoidal section



Most canals are excavated to a trapezoidal section, and this is often used as a convenient approximation to river cross-sections too. In many of the problems in this course we will consider the case of trapezoidal sections. Consider the quantities shown in the figure: the bottom width is W, the depth is h, the top width is B, and the *batter slope,* defined to be the ratio of H:V dimensions is m. Geometrically, $B = W + 2mh$, area $A = h\,(W + mh)$, wetted perimeter $P = W + 2\sqrt{1+m^2}\,h$.

Example 1 Calculate the normal depth in a trapezoidal channel of slope 0.001, $n = 0.04$, bottom width $W = 10\,\mathrm{m}$, with batter slopes $m = 2$, carrying a flow of $20\,\mathrm{m^3 s^{-1}}$. We have $A = h\,(10 + 2\,h)$, $P = 10 + 4.472\,h$. For B_0 we use $W = 10\,\mathrm{m}$. Equation (2.24) gives

$$h_0 = \left(\frac{nQ}{B_0\sqrt{S}}\right)^{3/5} = \left(\frac{0.04 \times 20}{10\sqrt{0.001}}\right)^{3/5} = 1.745\,\mathrm{m}.$$

Then, equation (2.23) gives

$$h_{i+1} = \left(\frac{nQ}{\sqrt{S_0}}\right)^{3/5} \times \frac{(10 + 4.472\,h_i)^{2/5}}{10 + 2\,h_i} = 6.948 \times \frac{(10 + 4.472\,h_i)^{2/5}}{10 + 2\,h_i}.$$

With $h_0 = 1.745$, we obtain $h_1 = 1.629$, $h_2 = 1.639$, $h_3 = 1.638\,\mathrm{m}$, and the method has converged.

3. Energy and momentum

We consider some of the simplest concepts of hydraulics with a critical view, and generalise them.

3.1 Energy and head

Consider the expression for the rate of transmission of energy E across a vertical cross-section (*e.g.* White 2009, §3.7) of area A:

$$E = \int_A \left(p + \rho\left(gz + \tfrac{1}{2}\left(u^2 + v^2 + w^2\right)\right)\right) u\,\mathrm{d}A\,, \tag{3.1}$$

in which p is pressure, its contribution actually being the rate of work being done by it, ρ is density, $gz + \frac{1}{2}\left(u^2 + v^2 + w^2\right)$ is the energy per unit mass where g is gravitational acceleration, z is the vertical co-ordinate, the velocity has horizontal component u along the channel, v and w normal to that; $\mathrm{d}A$ is an element of area of the cross-section.

This is an integral energy formulation, and is *not* Bernoulli's theorem (which is actually a momentum equation valid along a streamline (§3.5 White 2009) and is occasionally very useful such as determining the velocity upstream of a Pitot tube). It is actually simpler to use energy flux than Bernoulli.

Now we consider the individual contributions:

(a) Kinetic energy term

If the flow is swirling, then the v and w components will contribute, and if the flow is turbulent there will be extra contributions as well. Consider now the time mean value of the integral, denoted by an overbar

$$\frac{\rho}{2}\int_A \overline{\left(u^2+v^2+w^2\right)\,u}\,\mathrm{d}A,$$

and now if we substitute the velocity components $(u, v, w) = (\bar{u} + u', \bar{v} + v', w = \bar{w} + w')$, where overbars are time mean values and the $'$ show fluctuating values, we perform standard methods such as in §1.4.1 to expand the terms, using $\overline{u'} = \overline{v'} = \overline{w'} = 0$. The contribution can thus be written in terms of the mean streamwise component of velocity $U = Q/A$ as

$$\text{Kinetic contribution} = \alpha \times \tfrac{1}{2}\rho U^3 A, \tag{3.2}$$

where α is a dimensionless coefficient

$$\alpha = \frac{1}{U^3 A}\int_A \left(\bar{u}\left(\bar{u}^2 + \bar{v}^2 + \bar{w}^2 + 3\overline{u'^2} + \overline{v'^2} + \overline{w'^2} + \ldots\right)\right)\mathrm{d}A\,, \tag{3.3}$$

in terms of mean velocity components and mean squares of fluctuating components: α is termed the Coriolis energy coefficient, with a value slightly greater than 1. Traditionally, only the leading term in the integral has been used, with

$$\alpha = \frac{\int_A \bar{u}^3 \mathrm{d}A}{U^3 A}\,, \tag{3.4}$$

as obtained by Coriolis, a French engineer, who introduced it in the 1830s. It has been used in *all* writings since, except for Fenton (2005) who obtained equation (3.3) including the transverse velocity components and turbulent contributions, but the world is not interested. An estimate of the magnitudes of all the turbulent contributions can be had from Nezu & Nakagawa (1993), where equations (4.3-4.5) show that the term $3\overline{u'^2} + \overline{v'^2} + \overline{w'^2}$ is approximately $3.8\,\overline{u'^2}$. Ast $\overline{u'^2}$ is very roughly $0.01\bar{u}^2$ the relative contributions of the turbulent terms is something like 0.038. To allow for variation over the section, we approximate the velocity distribution by a $1/7$ power law, and use equation (3.4) to find the contribution of just the longitudinal velocity distribution, giving $\alpha \approx (1 + \frac{1}{7})^3/(1 + \frac{3}{7}) \approx 1.045$. Adding the turbulence contribution above gives $\alpha \approx 1.08$, at least. In a real stream, where there is greater variation of velocity over a section, α would be larger. For compound channels very much larger values may be encountered.

We have shown that there is something like a 10% correction because of turbulence and velocity varying over the section. However, usually we do not know enough information to evaluate it accurately. It is desirable to include this parameter in our work, which we will do, but often it will be a reminder that we are only obtaining an approximate solution. As such, it is useful!

(b) Pressure and potential head terms

In equation (3.1) these are combined as

$$\int_A (p + \rho g z)\, u\,\mathrm{d}A. \tag{3.5}$$

The approximation we now make, common throughout almost all open-channel hydraulics, is the

hydrostatic approximation, that pressure at a point of elevation z is given by

$$p \approx \rho g \times \text{ height of water above} = \rho g\,(\eta - z)\,, \tag{3.6}$$

where the free surface directly above has elevation η. This is the expression obtained in hydrostatics for a fluid which is not moving (equation 1.4). It is an excellent approximation in open channel hydraulics except where the flow is strongly curved, such as where there are short waves on the flow, or near a structure which disturbs the flow. Substituting equation (3.6) into equation (3.5) gives

$$\rho g \int_A \eta\, u\,\mathrm{d}A,$$

for the combination of the pressure and potential head terms. If we make the reasonable assumption that η is constant across the channel the contribution becomes simply

$$\rho g \eta \int_A u\,\mathrm{d}A = \rho g \eta Q, \tag{3.7}$$

from the definition of discharge Q.

(c) Combined terms

Substituting equations (3.2) and (3.7) into (3.1) we obtain for the rate of transmission of energy

$$E = \rho g Q\left(\eta + \frac{\alpha}{2g}\frac{Q^2}{A^2}\right), \tag{3.8}$$

which, in the absence of losses, would be constant along a channel. This energy flux across a face includes the mass flow rate ρQ. However that is constant along a channel and we can introduce the concept of the *Mean Total Head* H such that

$$H = \frac{\text{Energy flux}}{g \times \text{Mass flux}} = \frac{E}{g \times \rho Q} = \eta + \frac{\alpha}{2g}\frac{Q^2}{A^2}, \tag{3.9}$$

which has units of length and is easily related to elevation in many hydraulic engineering applications, relative to an arbitrary datum.

3.2 Momentum flux

The momentum flux across a vertical section is defined to be the sum of the pressure force, plus the mass rate of transport $\rho u\,\mathrm{d}A$ multiplied by the horizontal velocity u. The momentum flux is

$$M = \int_A \left(p + \rho u^2\right)\,\mathrm{d}A. \tag{3.10}$$

Substituting the hydrostatic pressure distribution, equation (1.4), we obtain

$$M = \rho\int_A \left(g\,(\eta - z) + u^2\right)\,\mathrm{d}A. \tag{3.11}$$

Now we evaluate this in terms of the quantities at the section.

Pressure contribution $\rho\int_A g(\eta - z)\,\mathrm{d}A$: The integral $\int_A (\eta - z)\,dA$ is simply the first moment of area about a transverse horizontal axis *at the surface*, we can write it as

$$\int_A (\eta - z)\,\mathrm{d}A = A\bar{h}, \tag{3.12}$$

where $\bar{h}$ is the *depth* of the centroid of the section *below the surface*.

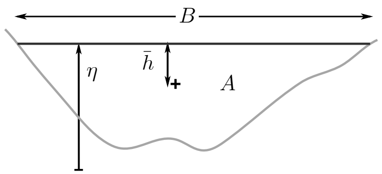


Figure 3.1: Cross-section of channel showing physical dimensions

Velocity contribution $\rho \int_A u^2\, dA$ **:** As with the kinetic energy integral, we include turbulence and allow for variation over the section, introducing a coefficient β which will be somewhat greater than unity, defined by

$$\begin{aligned} \beta &= \frac{1}{U^2 A} \int_A \overline{u^2}\,\mathrm{d}A \\ &= \frac{1}{U^2 A} \int_A \left(\bar{u}^2 + \overline{u'^2}\right)\mathrm{d}A. \end{aligned} \tag{3.13}$$

This coefficient is a Boussinesq momentum coefficient. Typical real values are $\beta = 1.05 - 1.15$. The effects of turbulence were added by Fenton (2005).

Collecting contributions, and substituting $U = Q/A$ we have the expression for the momentum flux at a section

$$M = \rho\left(gA\bar{h} + \beta\frac{Q^2}{A}\right). \tag{3.14}$$

4. Froude number

William Froude[1] (1810-1879) was a naval architect who proposed similarity rules for free-surface flows. A Froude number is a dimensionless number from a velocity scale U and a length scale L, $\mathsf{F} = U/\sqrt{gL}$. In the original definition, of a ship in deep water, the only length scale was L, the length of the ship. In river engineering it is not obvious what the length scale is. Might it be the wetted perimeter P, might it be the geometric mean depth A/B, where A is cross-sectional area and B is surface width?

Here we examine different aspects of the hydraulic Froude number. There are several facets.

Dimensional analysis of quantities at a section

If one conducts a dimensional analysis including the quantities Q, g, A, and B, we obtain the dimensionless number $\mathsf{F} = Q/\sqrt{gA^3/B}$, the traditional hydraulic definition. We have encountered the problem, common in dimensional analysis, that it has not told us what quantities we should include in the dimensional analysis, or at the end, to what power the quantity should be raised for physical significance. The lecturer remembers it in the form $\mathsf{F}^2 = Q^2B/gA^3$; others might think of it in the form $\mathsf{F} = U/\sqrt{gA/B} = U/\sqrt{gh}$, where h is mean depth.

Subjective visual classification – measure of wave-making ability

Every hydraulic engineer knows what a flow with a large sub-critical Froude number looks like,

[1] Pronounced as in "food". There is a charming article on the pronunciation by a famous USAmerican, Rouse (1965). One asks "Well, if Froude is "food", how is Rouse pronounced?". The answer is, apparently paradoxically, not "Roos" but as in "mouse". It's English.

with the possible presence of finite shorter waves on the surface, expressing proximity to critical. Similarly for a super-critical Froude number there are two-dimensional shock waves and striations. However, for all small and intermediate values of Froude number with a smooth free surface, one cannot estimate it visually. We all know what we mean when we refer to a "high Froude number flow", but it is subjective, and we have ignored the fact that wavemaking ability is proportional to the *square* of the Froude number.

Flows which are fast and shallow have large Froude numbers, and those which are slow and deep have small Froude numbers. Generally F^2 is an expression of the wave-making ability of a flow, and in conversation we usually use "high/ low Froude number" as an expression of how fast a flow is. For example, consider a river or canal which is $2\ \mathrm{m}$ deep flowing at $0.5\ \mathrm{ms}^{-1}$ (make some effort to imagine it - we can well believe that it would be able to flow with little surface disturbance!). We have

$$\mathsf{F} = \frac{U}{\sqrt{gh}} \approx \frac{0.5}{\sqrt{10 \times 2}} = 0.11 \quad \text{and} \quad \mathsf{F}^2 = 0.012\,,$$

and we can imagine that the wavemaking effects are small. Now consider flow in a street gutter after rain. The velocity might also be $0.5\ \mathrm{ms}^{-1}$, while the depth might be as little as $2\ \mathrm{cm}$. The Froude number is

$$\mathsf{F} = \frac{U}{\sqrt{gh}} \approx \frac{0.5}{\sqrt{10 \times 0.02}} = 1.1 \quad \text{and} \quad \mathsf{F}^2 = 1.2\,,$$

and we can easily imagine it to have many waves and disturbances on it due to irregularities in the gutter.

Relative importance of kinetic terms in governing equations

Now to consider the significance of Froude number we might look for it as something which expresses the ratio of kinetic energy to potential energy. Using Head does not provide assistance, as in Eqn (3.9), $H = \eta + \alpha Q^2/2gA^2$, η has an arbitrary origin. If we were to arbitrarily set the origin at the bottom of the channel ("Specific Energy") there is no advantage, as we obtain $H = h\left(1 + \alpha Q^2/2gA^2h\right) = h\left(1 + \frac{1}{2}\alpha\mathsf{F}^2 \times (A/B)/h\right)$. The kinetic term is proportional to F^2, but the presence of the factor $(A/B)/h$ (mean depth divided by maximum depth) shows that it cannot really be called a pure F^2 term. For years in his lectures the author tried to assert that this was the significance of Froude number, rather unconvincingly in retrospect.

If we consider momentum flux from equation (3.14) $M = \rho\left(gA\bar{h} + \beta Q^2/A\right)$ we find a similar result to head, giving $M = \rho gA\bar{h}\left(1 + \beta\mathsf{F}^2\,(A/B)/\bar{h}\right)$ and again the term containing F^2 contains another geometric ratio.

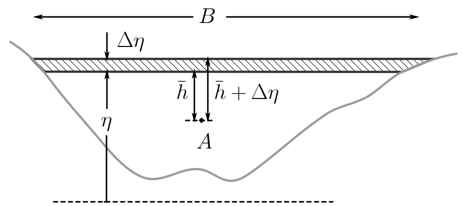


Figure 4.1: Cross-section of channel showing dimensions, the change of area due to an elemental surface height increase, and the depth of the centroid for the original and increased surface heights

Now we consider differential expressions. First we differentiate H with respect to surface elevation

$$\frac{\mathrm{d}H}{\mathrm{d}\eta}=\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta+\frac{\alpha Q^2}{2gA^2}\right)=1-\frac{\alpha Q^2}{gA^3}\frac{\mathrm{d}A}{\mathrm{d}\eta}.$$

Consider Figure 4.1 showing an elemental increase in surface elevation. It is clear that $\Delta A\approx B\,\Delta\eta$ and so in the limit, $\mathrm{d}A/\mathrm{d}\eta=B$, (a useful expression!) giving

$$\frac{\mathrm{d}H}{\mathrm{d}\eta}=1-\alpha\mathsf{F}^2. \tag{4.1}$$

Now we see F appearing alone (well, raised to the power 2 and with a coefficient of α) without any other geometric quantities and so we have a clear expression of the significance of Froude number: that if the surface elevation increases, the cross-sectional area increases, velocity decreases, with a relative change in the kinetic term of $-\alpha\mathsf{F}^2$.

Similarly for momentum: differentiating equation (3.14):

$$\frac{\mathrm{d}M}{\mathrm{d}\eta}=\rho g\left(\frac{\mathrm{d}}{\mathrm{d}\eta}\left(A\bar{h}\right)-\beta\frac{Q^2}{gA^2}\frac{\mathrm{d}A}{\mathrm{d}\eta}\right). \tag{4.2}$$

Considering Figure 4.1 we can calculate the change of $A\bar{h}$ as the surface increases by $\Delta\eta$. The surface is moved further from the original centroid by a distance $\Delta\eta$ giving a contribution to ΔM of $A\,\Delta\eta$. There is an extra contribution – the first moment of the incremental area is $B\,\Delta\eta\times\frac{1}{2}\Delta\eta$, but in the limit as $\Delta\eta\to 0$ this is vanishingly small and so we obtain $\mathrm{d}A\bar{h}/\mathrm{d}\eta=A$ (also useful!).

Substituting that and $\mathrm{d}A/\mathrm{d}\eta=B$ into equation (4.2) gives

$$\frac{\mathrm{d}M}{\mathrm{d}\eta}=\rho gA\left(1-\beta\mathsf{F}^2\right), \tag{4.3}$$

an expression equivalent to that from head, equation (4.1), and with a similar conclusion that $-\beta\mathsf{F}^2$ is the relative change in momentum flux for a change in surface elevation.

What is interesting is that nowhere in these notes does F^1 appear, instead, only F^2. One could also say that F^2 itself has no significance in the dynamic equations without being written as either $\alpha\mathsf{F}^2$ or $\beta\mathsf{F}^2$, the relative contributions of the kinetic term in differential relations. Even if we were to consider rather more complicated problems such as the unsteady propagation of waves and floods, and to non-dimensionalise the equations, we would find that the Froude number F itself never appears in the equations, but always as $\alpha\mathsf{F}^2$ or $\beta\mathsf{F}^2$, depending on whether energy or momentum considerations are being used.

Attempt to express F in terms of wave speed

In the literature F has often been written as $\mathsf{F}=U/c$, the ratio of fluid speed to the supposed long wave speed $c=\sqrt{gA/B}$. Whereas making that substitution does give the value of F obtained above by dimensional analysis, it is not much help when looking at a flowing stream, to consider what the wave speed is and what is the fluid velocity relative to that. If one were seeking a precise mathematical definition, it is still not the answer. The speed of waves according to the long wave equations depends on the period of the waves (Fenton 2015*a*, §1.5), with shorter waves actually travelling faster, and there is no such thing as a unique long wave speed. In the limit of the longest

waves, the equations dominated by resistance, the wave speed is approximately $c=\frac{3}{2}U$. In the other limit, that least affected by resistance, where the waves are relatively short but the long wave equations are still satisfied, the speed is $c=\sqrt{gA/B+(\beta^2-\beta)U^2}$, and if we consider $\beta=1$ as a (quite good!) approximation we recover the traditional result $c=\sqrt{gA/B}$, but we could not claim it as a fundamental property.

F^2 as determined by a ratio of bed slope to dimensionless resistance term OR F^2 is almost constant for any one stream

There is a simple expression for F^2 that we can obtain based on common uniform flow formulae which provides some insight as to its magnitude, if not its effects. The Chézy-Weisbach flow formula is

$$U=\frac{Q}{A}=\sqrt{\frac{8g}{\lambda}\frac{A}{P}S}=\sqrt{\frac{g}{\Lambda}\frac{A}{P}S}, \tag{4.4}$$

where λ is the Weisbach resistance coefficient, P is the wetted perimeter, and S is slope . This can trivially be re-written to give

$$\mathsf{F}^2=\frac{U^2}{gA/B}=\frac{S}{\Lambda}\frac{B}{P}\approx\frac{S}{\Lambda}\quad\text{for a wide channel.} \tag{4.5}$$

Using the Gauckler-Manning flow formula instead gives the result as

$$\mathsf{F}^2=\frac{S}{gn^2\left(P/A\right)^{1/3}}\frac{B}{P}\approx\frac{S}{gn^2\left(P/A\right)^{1/3}}\quad\text{for a wide channel,} \tag{4.6}$$

expressed in terms of the dimensionless resistance quantity $gn^2\left(P/A\right)^{1/3}$, a slowly-varying function of A/P.

These expressions show that F^2 is given by a ratio of bed slope to a dimensionless resistance term, which is obvious in retrospect, given the nature of the two flow formulae. This means that for a particular reach of river, where slope S is effectively independent of flow, where B/P also does not vary much with the flow and Λ often does not vary much, the Froude number F does not change much with flow. While a flood might look more dramatic than a more-common low flow, because it is faster and higher, the Froude number is roughly the same for both.

One wonders if it might not help in future to consider F^2 not as a "velocity-squared-divided-by-g-and-mean-depth", but as a ratio of bed slope to dimensionless resistance?

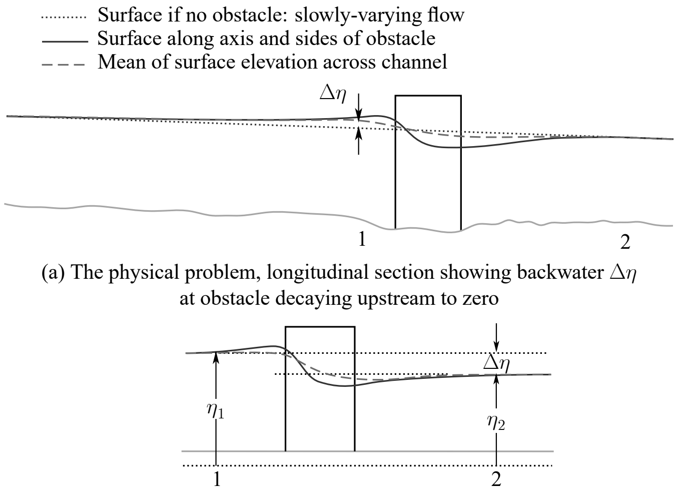
5. The effect of obstructions on streams – an approximate method

The main objective here is to obtain a convenient theory for the effect on a stream of a partial obstruction. To do this we linearise the governing momentum equation. This is an example of how approximating a problem can give more insight and understanding.

River Traun, Bad Ischl, Oberösterreich

Structures such as weirs can almost completely block a river, but there are also other types of obstacles that are only a partial blockage, such as the piers of a bridge, blocks on the bed, Iowa vanes, *etc.* or possibly more importantly, the effects of trees placed in rivers ("Large Woody Debris"), used in their environmental rehabilitation. It might be important to know what the forces on the obstacles are, or, more importantly for us, what effects the obstacles have on the river. Here we set up the problem in conventional open channel theoretical terms. Then, however, we obtain an analytical solution by making the approximation that the effects of the obstacle are small. This gives us more insight into the nature and importance of the problem.

The physical problem and its idealisation



(a) The physical problem, longitudinal section showing backwater $\Delta\eta$ at obstacle decaying upstream to zero

(b) The idealised problem, uniform channel with no friction or slope

Figure 5.1: A typical physical problem of flow past a bridge pier, and its idealisation for hydraulic purposes

Momentum conservation

Consider equation (3.14) for the momentum flux:

$$M = \rho\left(gA\bar{h} + \beta\frac{Q^2}{A}\right). \tag{5.1}$$

if a force P is applied in a negative direction to a flow between two sections 1 and 2:

$$P = \rho\left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_1 - \rho\left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_2, \tag{5.2}$$

Usually one wants to calculate the effect of the obstacle on water levels. The effects of drag can be estimated by knowing the area of the object measured transverse to the flow, a, the drag coefficient C_{D}, and u, the mean fluid speed past the object:

$$P = \frac{1}{2}\rho C_{\mathrm{D}} u^2 a, \tag{5.3}$$

and so, substituting into equation (5.2) gives, after dividing by density,

$$\frac{1}{2}C_{\mathrm{D}} u^2 a = \left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_1 - \left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_2. \tag{5.4}$$

We consider the velocity u on the obstacle as being proportional to the upstream velocity, such that we write

$$u^2 = \Gamma\left(\frac{Q}{A_1}\right)^2, \tag{5.5}$$

where Γ is a coefficient which recognises that the velocity which impinges on the object is generally not equal to the *mean* velocity in the flow. For a small object near the bed, Γ could be quite small; for an object near the surface it will be slightly greater than 1; for objects of a vertical scale that of the whole depth, $\Gamma \approx 1$. Equation (5.4) becomes

$$\frac{1}{2}\Gamma\, C_{\mathrm{D}}\frac{Q^2}{A_1^2}a = \left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_1 - \left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_2 \tag{5.6}$$

A typical problem is where the downstream water level is given (sub-critical flow, so that the control is downstream), and we want to know by how much the water level will be raised upstream if an obstacle is installed. As both A_1 and $\bar{h}_1$ are functions of h_1, so that we would need to know in detail the geometry of the stream, and then to solve the transcendental equation for h_1. However, by *linearising* the problem, solving it approximately, we obtain a simple explicit solution that tells us rather more.

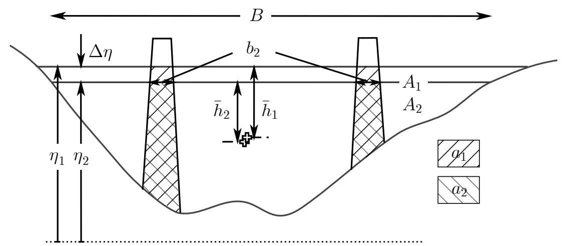


Figure 5.2: Cross-section showing dimensions for water levels at 1 and 2

Consider the stream cross-section shown in Fig. 5.2, with a small change in water level $\eta_1 = \eta_2 + \Delta\eta$. We now use geometry to obtain approximate expressions for quantities at 1 in terms of those at 2. On Page 49 we saw that $\Delta A \approx B\,\Delta\eta$ and $\Delta\left(A\bar{h}\right) \approx A\,\Delta\eta$, so that

$$A_1 \approx A_2 + B_2\,\Delta\eta, \text{ and } \left(A\bar{h}\right)_1 \approx \left(A\bar{h}\right)_2 + A_2\,\Delta\eta\ ,$$

and similarly we write for the blockage area $a_1 \approx a_2 + b_2\,\Delta\eta$, where b_2 is the surface width of the obstacle (which for a submerged obstacle would be zero).

The momentum equation (5.6) gives us

$$\frac{1}{2}\Gamma\, C_{\mathrm{D}} \frac{Q^2}{A_1^2} a_2 = gA_2\,\Delta\eta + \beta\frac{Q^2}{A_2 + B_2\,\Delta\eta} - \beta\frac{Q^2}{gA_2}. \tag{5.7}$$

Now we use the binomial theorem with an expansion in $\Delta\eta$ to linearise the equation

$$\frac{1}{A_2 + B_2\,\Delta\eta} = \frac{1}{A_2\left(1 + B_2\,\Delta\eta/A_2\right)} = \frac{1}{A_2}\left(1 + B_2\,\Delta\eta/A_2\right)^{-1} \approx \frac{1}{A_2}\left(1 - B_2\,\Delta\eta/A_2\right),$$

neglecting terms like $(\Delta\eta)^2$. Equation (5.7) becomes

$$\frac{1}{2}\Gamma\, C_{\mathrm{D}} \frac{Q^2}{gA_1^2} a_2 \approx \Delta\eta\, A_2\left(1 - \beta\frac{Q^2 B_2}{gA_2^3}\right).$$

As a_2 on the left and $\Delta\eta$ on the right are small we can use $A_1 = A_2 + O\left(\Delta\eta\right)$ to replace A_1 by A_2 and introduce the symbol $\mathsf{F}_2^2 = Q^2 B_2/gA_2^3$, the square of the Froude number of the downstream flow. The equation is then easily solved for $\Delta\eta$ to give an explicit approximation for the increase in upstream depth caused by the obstacle expressed as $\Delta\eta/\left(A_2/B_2\right)$, where A_2/B_2 is the mean

downstream depth:

$$\frac{\Delta\eta}{A_2/B_2} = \frac{\frac{1}{2}\Gamma\, C_{\mathrm{D}}\, \mathsf{F}_2^2}{1 - \beta\mathsf{F}_2^2}\frac{a_2}{A_2}, \tag{5.8}$$

expressed only in terms of known downstream quantities. This explicit approximate solution has revealed the important quantities of the problem to us and how they affect the result:

- Relative blockage area a_2/A_2, and
- Froude number $\mathsf{F}_2^2 = Q^2 B_2/gA_2^3$. For subcritical flow $\beta\mathsf{F}_2^2 < 1$ the denominator in (5.8) is positive, and so is $\Delta\eta$, so that the surface drops from 1 to 2, as we expect. If the flow is supercritical, $\beta\mathsf{F}_2^2 > 1$, we find $\Delta\eta$ negative, and the surface rises between 1 and 2. If the flow is near critical $\beta\mathsf{F}_2^2 \approx 1$, the change in depth will be large, which is made explicit by the theory, and in that case it is not valid.

We can immediately estimate the effect of an obstacle. We see that, for small Froude number $\mathsf{F}_2^2 \ll 1$, such that $1 - \beta\mathsf{F}_2^2 \approx 1$, the relative change of depth is equal to $\frac{1}{2}$ times $\Gamma \approx 1$ (for a body extending the whole depth), times $C_{\mathrm{D}} \approx 1$ for cylinders *etc*., multiplied by F_2^2, usually small, multiplied by the blockage ratio a_2/A_2, which is also probably small. So, approximately $\frac{1}{2}\mathsf{F}^2 a/A$, the relative result, which is usually small. However, the absolute value might still be finite compared with resistance losses, as will be seen below.

Another benefit of the approximate analytical solution is that it shows that such an obstacle forms a control in the channel, so that the finite sudden change in surface elevation $\Delta\eta$ is a function of Q^2, or Q a function of $\sqrt{\Delta\eta}$, in a manner analogous to a weir. In numerical river models it should

ideally be included as an internal boundary condition between different reaches as if it were a type of fixed control.

The mathematical step of linearising has revealed much to us about the nature of the problem that the original momentum equation did not.

Example 2 It is proposed to build a bridge, where the bridge piers occupy about 10% of the "wetted area" of a river with Froude number 0.5 (which is quite large). How much effect will this have on the river level upstream?

As the bridge piers occupy all the depth, we have $\Gamma = 1$. A typical drag coefficient is $C_{\mathrm{D}} \approx 1$. We will use $\beta = 1$ (this is an estimate!). So we find, using equation (2.23):

$$\begin{aligned}\frac{\Delta\eta}{A_2/B_2} &= \frac{\frac{1}{2}\Gamma\, C_{\mathrm{D}}\, \mathsf{F}_2^2}{1 - \beta\mathsf{F}_2^2}\frac{a_2}{A_2} \approx \tfrac{1}{2} \times 1 \times 1 \times \frac{0.5^2}{1 - 0.5^2} \times 0.1 \\ &= 0.017,\end{aligned}$$

about 2% of the mean depth. This seems small, but if the river were $2\ \mathrm{m}$ deep, there is a $4\ \mathrm{cm}$ drop across the bridge. If the slope of the river were $S = 10^{-4}$, this would correspond to the surface level change in a length of $400\ \mathrm{m}$, which can hardly be neglected. In most rivers F is rather smaller than this, and the effect is small. We might have saved ourselves an expensive laboratory program.

Generalising to compound structures

In the presentation of the theory we implicitly declared a_1 and a_2 to be the *total* blockage area,

upstream and downstream. In the case of the two similar piers in Figure 5.2 it was simpler not to complicate the presentation any more. Here we note that a general result is possible for compound structures such as, say, a bridge with multiple, possibly dissimilar, piers, plus a deck, plus railings on the deck, *etc*. As the force contributions are additive, we can simply generalise equation (5.8) for the total change of water level

$$\frac{\Delta\eta}{A_2/B_2} = \frac{\frac{1}{2}\mathsf{F}_2^2}{1 - \beta\mathsf{F}_2^2}\sum_i \Gamma_i\, C_{\mathrm{D},i}\frac{a_{2,i}}{A_2}, \tag{5.9}$$

summing in i over all the parts of the structure, and where the different Γ_i reflect the magnitude of the fluid velocity for each component: for the bridge piers over the whole depth the mean velocity is approximately the mean velocity in the stream so $\Gamma_i \approx 1$; for the bridge deck the surface velocity is greater than the mean, so $\Gamma_i > 1$, and so on.

A simple estimate of the effect of multiple obstacles on a river

If we want to approximate the effect of a number of obstacles such as a number of different bridges or "large woody debris" we can take equation (5.9) and dropping the subscripts 2 such that A *etc.* just represent typical values in the river, dividing by the length L of the reach of the river,

$$\frac{\Delta\eta}{L} = \frac{1}{LB}\frac{\frac{1}{2}\mathsf{F}^2}{1 - \beta\mathsf{F}^2}\sum_i \Gamma_i\, C_{\mathrm{D},i}\, a_i \tag{5.10}$$

but this is the contribution to the slope of the river surface due to the obstacles. If we substitute this for slope S in the Weisbach equation (2.18) and re-arrange, writing $\mathsf{F}^2 = U^2/g\left(A/B\right)$ we obtain an expression for the contribution of the obstacles to the resistance coefficient Λ:

$$\Lambda = \tfrac{1}{2}\frac{1}{PL}\frac{1}{1-\beta\mathsf{F}^2}\sum_i \Gamma_i\, C_{\mathrm{D},i} a_i \;.$$

Hence the effect of a number of obstacles on a stream, such as woody debris could be simply and approximately incorporated in a continuous model of a stream.

Practical warning

Don't use the popular software HEC-RAS (2016) to solve such bridge problems – it makes elementary *and complicating* hydraulic mistakes, such as assuming that the suddenly-varied flow past a bridge pier is treated as a gradual channel contraction. It is quite all right to do as we have done here, to treat it as if it were a bluff object. It is.

6. Reservoir routing

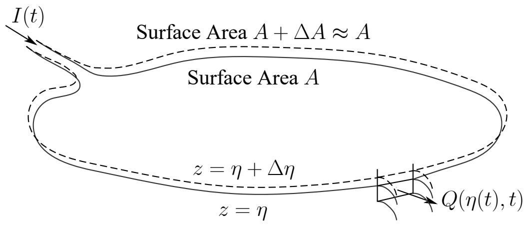


Figure 6.1: Reservoir or tank, showing surface level varying with inflow, determining the rate of outflow

Consider the problem shown in Figure 6.1, where a generally unsteady inflow rate $I(t)$ enters a reservoir or a storage tank, and we have to calculate what the outflow rate $Q(t)$ is, as a function of time t. The action of the reservoir is usually to store water, and to release it more slowly, so that the outflow is delayed and the maximum value is less than the maximum inflow. Some reservoirs, notably in urban areas, are installed just for this purpose, and are called *detention* reservoirs or storages. The procedure of solving the problem is also called *Level-pool Routing*.

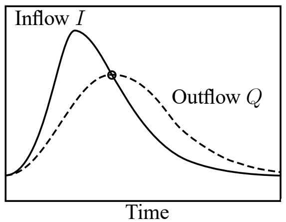


Figure 6.2:

The process is shown in Figure 6.2. When a flood comes down the river, inflow increases, the water level rises in the reservoir until at the point O when the outflow over the spillway now balances the inflow. At this point, outflow and surface elevation in the reservoir have a maximum. After this, the inflow might reduce quickly, but it still takes some time for the extra volume of water to leave the reservoir.

Detention reservoir in a public park in Melbourne, Australia

It is simple and obvious to write down the relationship stating that the rate of surface rise $\mathrm{d}\eta/\mathrm{d}t$ is equal to the net rate of volume increase divided by surface area:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{I(t) - Q(\eta, t)}{A(\eta)}\,, \tag{6.1}$$

where η is the free surface elevation, and $A(\eta)$ is the surface area, possibly given from planimetric information from contour maps, and $Q(\eta, t)$ is the volume rate of outflow, which is usually a simple function of the surface elevation η, from a weir or gate formula, usually involving terms like $(\eta - z_{\text{outlet}})^{1/2}$ and/or $(\eta - z_{\text{crest}})^{3/2}$, where z_{outlet} is the elevation of the pipe or tailrace outlet to atmosphere and z_{crest} is the elevation of the spillway crest. There might be extra dependence on time t if the outflow device is opened or closed. This is a differential equation for the surface elevation itself. The procedure of solving it is called *Level-pool Routing*.

The traditional method of solving the problem, described in almost all books on hydrology, is to use an unnecessarily complicated method called the "Modified Puls" method of routing, which solves a transcendental equation for a single unknown quantity, the volume in the reservoir, at each time step. It is simpler and more fundamental to treat the problem as a differential equation (Fenton 1992). :-)

Numerical solution of the differential equation by Euler's method

Euler's method is the simplest (but least-accurate) of all methods, being of first-order accuracy only. For river engineering purposes it is usually quite good enough. However there is a good method for making it more accurate, which we will use. Euler's method is to approximate the derivative in a differential equation at a time step i by a forward difference expression in terms of a time step Δ, here applying it to equation (6.1):

$$\left.\frac{\mathrm{d}\eta}{\mathrm{d}t}\right|_i \approx \frac{\eta_{i+1} - \eta_i}{\Delta} = \frac{I(t_i) - Q(\eta_i, t_i)}{A(\eta_i)}\,,$$

giving the scheme to calculate the value of η at t_{i+1} as

$$\eta_{i+1} = \eta_i + \Delta\frac{I(t_i) - Q(\eta_i, t_i)}{A(\eta_i)} + O\left(\Delta^2\right), \tag{6.2}$$

where we use the notation η_i for the solution at time step i. We have shown that the error of this approximation is proportional to Δ^2. It is necessary to take small enough Δ that this is small.

Accurate results with simple methods – Richardson extrapolation

We introduce a clever device for obtaining more accurate solutions from Euler's method and others.

Consider the numerical value of any part of a computational solution for some physical quantity ϕ obtained using a time or space step Δ, such that we write $\phi(\Delta)$. Let the computational scheme be of known nth order such that the *global* error of the scheme at any point or time is proportional to

Δ^n, then if $\phi(0)$ is the exact solution, we can write the expression in terms of the error at order n:

$$\phi(\Delta) = \phi(0) + b\Delta^n + \dots\ , \tag{6.3}$$

where $\phi(0)$ is the solution for a vanishingly small time step, so that it should be exact. The b is an unknown coefficient; the neglected terms vary like Δ^{n+1}. If we have two numerical simulations or approximations with two different Δ_1 and Δ_2 giving numerical values $\phi_1 = \phi(\Delta_1)$ and $\phi_2 = \phi(\Delta_2)$ then we write (6.3) for each:

$$\begin{aligned} \phi_1 &= \phi(0) + b\Delta_1^n + \dots\ , \\ \phi_2 &= \phi(0) + b\Delta_2^n + \dots\ . \end{aligned}$$

These are two linear equations in the two unknowns $\phi(0)$ and b. Eliminating b, which is not important, between the two equations and neglecting the terms omitted, we can solve for $\phi(0)$, an approximation to the exact solution:

$$\phi(0) = \frac{\phi_2 - r^n\phi_1}{1 - r^n} + O\left(\Delta_1^{n+1}, \Delta_2^{n+1}\right)\ , \tag{6.4}$$

where $r = \Delta_2/\Delta_1$. The errors are now proportional to step size to the power $n+1$, so that we have gained a higher-order scheme without having to implement any more sophisticated numerical methods, just with a simple numerical calculation. This procedure, where n is known, is called *Richardson extrapolation to the limit*.

1. For simple Euler time-stepping solutions of ordinary differential equations, $n = 1$, and if we

perform two simulations, one with a time step Δ and then one with $\Delta/2$, we have

$$\phi(t, 0) = 2\,\phi(t, \Delta/2) - \phi(t, \Delta) + O\left(\Delta^2\right), \tag{6.5}$$

where the numerical solution at time t has been shown as a function of the step. This is very simply implemented.

2. For the evaluation of an integral by the trapezoidal rule, $n = 2$.

Example 3 Example 4 Consider a small detention reservoir, square in plan, with dimensions 100m by 100m, with water level at the crest of a sharp-crested weir of length of $b = 4\,\mathrm{m}$, where the outflow over the sharp-crested weir can be taken to be

$$Q\left(\eta\right) = 0.6\sqrt{g}b\eta^{3/2}, \tag{6.6}$$

where $g = 9.8\,\mathrm{ms}^{-2}$. The surrounding land has a slope (V:H) of about 1:2, so that the length of a reservoir side is $100 + 2 \times 2 \times \eta$, where η is the surface elevation relative to the weir crest, and

$$A(\eta) = (100 + 4\eta)^2\,.$$

The inflow hydrograph is:

$$I(t) = Q_{\text{min}} + (Q_{\text{max}} - Q_{\text{min}})\left(\frac{t}{T_{\text{max}}}e^{1-t/T_{\text{max}}}\right)^5, \tag{6.7}$$

where the event starts at $t = 0$ with Q_{min} and has a maximum Q_{max} at $t = T_{\text{max}}$. This general form of inflow hydrograph mimics a typical storm, with a sudden rise and slower fall, and will be used in other places in this course. In the present example we consider a typical sudden local storm event,

with $Q_{\text{min}} = 1\ \mathrm{m^3s^{-1}}$ at $T_{\text{max}} = 1800\,\mathrm{s}$.

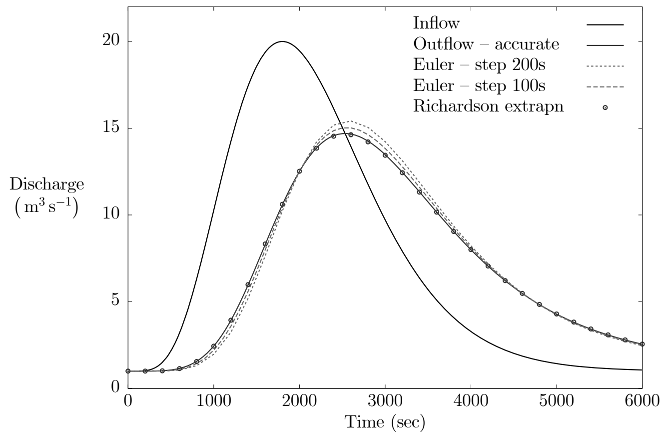


Figure 6.3: Computational results for the routing of a sudden storm through a small detention reservoir

The problem was solved with an accurate 4th-order Runge-Kutta scheme, and the results are shown as a solid blue line on figure 6.3, to provide a basis for comparison. Next, Euler's method (equation 6.2) was used with 30 steps of $200\,\mathrm{s}$, with results that are barely acceptable. Halving the time step to $100\,\mathrm{s}$ and taking 60 steps gave the slightly better results shown. It seems, as expected from knowledge of the behaviour of the global error of the Euler method, that it has been halved at each

point. Next, applying Richardson extrapolation, equation (6.5), gave the results shown by the solid points. They almost coincide with the accurate solution, and cross the inflow hydrograph with an apparent horizontal gradient, as required, whereas the less-accurate results do not. Overall, it seems that the simplest Euler method can be used, but is better together with Richardson extrapolation. In fact, there was nothing in this example that required large time steps – a simpler approach might have been just to take rather smaller steps.

The role of the detention reservoir in reducing the maximum flow from $20\ \mathrm{m^3s^{-1}}$ to $14.7\ \mathrm{m^3s^{-1}}$ is clear. If one wanted a larger reduction, it would require a larger spillway. It is possible in practice that this problem might have been solved in an inverse sense, to determine the spillway length for a given maximum outflow.

7. The one-dimensional equations of river hydraulics

These are the fundamental equations that are used to describe the propagation of floods and disturbances in rivers. They are called the *long wave equations*, the *shallow water equations*, or the *Saint-Venant equations*, and are mass and momentum conservation equations for water.

The equations are a pair of partial differential equations in the independent variables x (distance along the stream) and t (time). A typical flood routing problem is for large extra values of discharge Q to be introduced at the upstream initial point, and then for a number of time steps, to solve the equations along the channel to obtain the progress of the flood at each time.

We will also consider a mass conservation equation for soil. In their steady form, the equations describe how water level and velocities vary along a stream, and what effects boundary changes such as sand removal might have on flooding.

We make the traditional approximation that all rivers are straight. Later we will see that it is quite accurate, even for meandering streams.

The model is one-dimensional. We do not consider details of motion in the plane across the stream – all quantities are averaged across it. This does not mean that we assume they are constant. This approach requires surprisingly few approximations – the model is a good simple model of complicated reality.

It is easier to use cartesian co-ordinates, for which we use x the horizontal distance along the stream, y the horizontal transverse co-ordinate, and z the vertical, relative to some arbitrary origin.

7.1 Mass conservation of water and soil

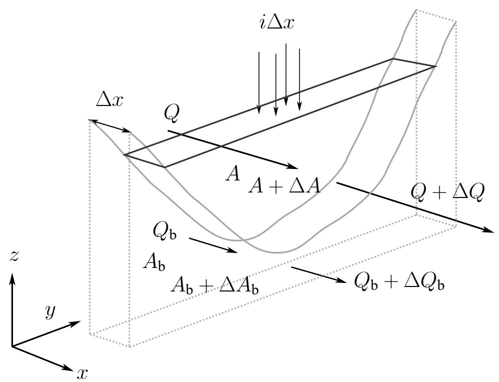


Figure 7.1: Elemental length of channel showing control volumes

Consider Figure 7.1, showing an elemental slice of channel of length Δx with two stationary vertical faces across the flow. It includes two different control volumes. The free surface and the interface between them may move. The surface shown by solid lines contains water and possibly suspended soil grains. The surface shown by dotted lines contains the soil moving as bed load and extends down into the soil such that there is no motion at its far boundaries. Each is modelled separately. We assume that the density of the fluid (water plus suspended soil particles) is constant, so that we can consider *volume* conservation.

On the upstream vertical face at any instant, there is a volume flux (rate of volume flow) Q, and that on the downstream face is $Q+\Delta Q$, so that

$$\text{Net volume flow rate of fluid } \textit{leaving} \text{ across vertical faces} = \Delta Q = \frac{\partial Q}{\partial x}\Delta x + \text{terms in } (\Delta x)^2 .$$

If rainfall, seepage, or tributaries contribute an inflow volume rate i per unit length of stream, the

volume flow rate of this other fluid *entering* the control volume is $i\,\Delta x$. If the sum of the two contributions is not zero, then volume of fluid is changing inside the elemental slice, so that the water level will change in time. The rate of change with time t of fluid volume is $\partial A/\partial t \times \Delta x$. For volume to be conserved (mass, but we assume the water is incompressible) this is equal to the net rate of fluid entering the control volume, dividing by Δx and taking the limit as $\Delta x \to 0$ gives

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = i. \tag{7.1}$$

This is the *mass conservation equation*. Remarkably for hydraulics, it is almost exact. The only approximation has been that the channel is straight. It is also linear in the two dependent variables A and Q.

The composite bulk density ρ_b of the bed load composed of larger soil particles and is assumed to be constant. The bed has a cross-sectional area A_b, the bulk volumetric flow rate is Q_b, and there is an inflow of mass rate $\dot{m}_i$ per unit length, possibly due to deposition or erosion. Mass conservation is calculated following the same reasoning as for the channel, giving Exner's[2] equation:

$$\frac{\partial A_\mathrm{b}}{\partial t} + \frac{\partial Q_\mathrm{b}}{\partial x} = \frac{\dot{m}_i}{\rho_\mathrm{b}}. \tag{7.2}$$

The volume transport rate used here is the bulk flow rate; it is related to the volume flow rate of solid matter Q_s used in transport formulae, by $Q_\mathrm{s} = Q_\mathrm{b}\,(1-\varphi)$, where φ is the porosity.

[2] `https://de.wikipedia.org/wiki/Felix_Maria_von_Exner-Ewarten` – Austrian – Director of the Zentral Anstalt für Meteorologie und Geodynamik

Upstream Volume

The mass conservation equation (7.1) suggests the introduction of a function $V(x,t)$ which is the volume upstream of point x at time t, such that for the channel flow

$$\frac{\partial V}{\partial x} = A \quad \text{and} \quad \frac{\partial V}{\partial t} = \int^x i\,(x')\;\mathrm{d}x' - Q. \tag{7.3}$$

The derivative of volume with respect to distance x gives the area, as shown, while the time rate of change of volume upstream is given by the rate at which the volume is increasing due to inflow, minus the rate at which volume is passing the point and therefore no longer upstream. Substituting for A and Q into equation (7.1):

$$\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial x}\left(-\frac{\partial V}{\partial t}\right) + \frac{\partial}{\partial x}\int^x i\,(x')\;\mathrm{d}x' = i\ .$$

The order of differentiation in the first two terms on the left does not matter, and they cancel. The derivative of the integral of $i\,(x')$ on the left is simply $i(x)$, the value on the right, and the equation is identically satisfied. Hence, by introducing V we automatically satisfy one of the two conservation equations and reduce the number of unknowns from two (A and Q) to one (V). Sometimes this can be very helpful.

In the case of the bed load, a similar quantity $V_\mathrm{b}(x,t)$ can be introduced such that the mass conservation equation (7.2) is identically satisfied.

Use of surface elevation instead of cross-sectional area

We usually work in terms of water surface elevation ("stage") η, which is easily measurable and which is practically more important. We make a significant assumption here, but one that is usually accurate: the water surface is horizontal across the stream. Now, if the surface changes by an amount $\delta\eta$ in an increment of time δt, then the area changes by an amount $\delta A = B\,\delta\eta$, where B is the width of the stream surface. Taking the usual limit of small variations in calculus, we obtain $\partial A/\partial t = B\,\partial\eta/\partial t$, and the mass conservation equation can be written

$$B\frac{\partial\eta}{\partial t} + \frac{\partial Q}{\partial x} = i. \tag{7.4}$$

The discharge Q could be written as $Q = UA$, where U is the mean streamwise velocity over a section, and substituted into this. However, the discharge is more practical and fundamental than the velocity, and that will not be done here.

7.2 Momentum conservation equation for channel flow

The equation

The conservation of momentum principle is now applied to the mixture of water and suspended solids in the main channel for a moving and deformable control volume (White 2003, §§3.2 &

3.4). The x-component is

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathrm{CV}} \rho u\,\mathrm{d}V}_{\text{Unsteady term}} + \underbrace{\int\limits_{\mathrm{CS}} \rho u\,\mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S}_{\text{Fluid inertia term}} = P_x, \tag{7.5}$$

where $\mathbf{u}$ is the fluid velocity with x-component u, $\mathrm{d}V$ is an element of volume, $\mathbf{u}_{\mathrm{r}}$ is the velocity vector of the fluid relative to the local element of the control surface, which is possibly moving itself, $\hat{\mathbf{n}}$ is a unit vector with direction normal to and directed outwards, and $\mathrm{d}S$ is an elemental area of the surface. The quantity $\mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}}$ is the component of relative velocity normal to the surface at any point. It is this velocity that is responsible for the transport of any quantity across the surface, momentum here. P_x is the force exerted on the fluid in the control volume by both body forces, which act on all fluid particles, and surface forces which act only on the control surface.

Hydraulic approximations

1. **Unsteady term**

   The element of volume is $\mathrm{d}V = \Delta x\,\mathrm{d}A$, and the term contribution can be written

   $$\rho\,\Delta x\,\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_A u\,\mathrm{d}A = \rho\,\Delta x\,\frac{\partial Q}{\partial t}, \tag{7.6}$$

   where the integral $\int_A u\,\mathrm{d}A$ has a simple and practical significance – it is just the discharge Q, so that the contribution of the term can be written simply as shown, but where again it has been necessary to use the partial differentiation symbol, as Q is a function of x as well. No additional

   approximation has been made in obtaining this term. It can be seen that the discharge Q plays a simple role in the momentum of the flow.

2. **Fluid inertia term**

   The second term on the left of equation (7.5) is $\int\limits_{\mathrm{CS}} \rho u\,\mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$, has its most important contributions from the stationary vertical faces perpendicular to the main flow.

   a. **Top and bottom, possibly moving surfaces:** we have chosen our control surface to coincide with these boundaries so that no fluid crosses them, $\mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}} = 0$ and there is no contribution.

   b. **Stationary vertical faces:** on the upstream face, $\mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}} = -u$, giving the contribution to the term of $-\rho\int_A u^2\,\mathrm{d}A$. The downstream face at $x + \Delta x$ has a contribution of a similar nature, but positive, and where all quantities have increased over the distance Δx. The net contribution, the difference between the two, after neglecting terms like $(\Delta x)^2$, can be written

   $$\rho\,\Delta x\,\frac{\partial}{\partial x}\int\limits_A u^2\,\mathrm{d}A.$$

   In equation (3.13), much earlier, we approximated the integral over the cross section and with a mean in time, in terms of a Boussinesq coefficient β such that the contribution to the equation is then simply but approximately.

   $$\rho\,\Delta x\,\frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right). \tag{7.7}$$

   It is useful to retain the β, unlike many presentations that implicitly assume it to be 1.0, as it

   is a signal and reminder to us that we have introduced an approximation.

   c. **Lateral momentum contributions:** If there is also fluid entering or leaving from rainfall, tributaries, or seepage, there are contributions over the other faces. Their contribution to momentum exchange is small and uncertain and we will ignore them.

3. **Contributions to force P_x**

   a. **Body force**: for the straight channel considered, the only body force acting is gravity; we will consider only the x-component of the momentum equation, which have chosen to be horizontal, as gravity only has a component in the $-z$ direction, there will be no contribution from gravity to our equation! The manner in which gravity acts is to cause pressure gradients in the fluid, giving rise to the following term, due to pressure variations around the control surface.

   b. **Pressure forces**: these act normally to the control surface. The direction of the pressure force on the fluid at the control surface is $-\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the outward-directed normal; its local magnitude is $p\,\mathrm{d}S$, where p is the pressure and $\mathrm{d}S$ an elemental area of the control surface. Hence, the total pressure force is the integral $-\int_{\mathrm{CS}} p\hat{\mathbf{n}}\,\mathrm{d}S$. This is difficult to evaluate for arbitrary control surfaces, as the pressure and the non-constant unit vector have to be integrated over all the faces. A considerably simpler derivation is obtained if the term is evaluated using the Divergence Theorem:

   $$-\int\limits_{\mathrm{CS}} p\,\hat{\mathbf{n}}\,\mathrm{d}S = -\int\limits_{\mathrm{CV}} \nabla p\,\mathrm{d}V,$$

where $\nabla p = (\partial p/\partial x, \partial p/\partial y, \partial p/\partial z)$, the vector gradient of pressure. This has turned a complicated surface integral into a volume integral of a simpler quantity.

We only need the x component $-\int_{\mathrm{CV}} \partial p/\partial x \, \mathrm{d}V$, the volume integral of the streamwise pressure gradient. The hydraulic approximation now has a problem, because we have not attempted to calculate the detailed pressure distribution throughout the flow. However, in most places in most channel flows the length of disturbances is much greater than the depth, so that streamlines in the flow are only very gently sloping and gently curved, and the pressure in the fluid is accurately given by the equivalent hydrostatic pressure, that due to a stationary column of water of the same depth. Hence we write for a point of elevation z, our equation (3.13) gives

$$p = \rho g \times \text{Depth of water above point} = \rho g(\eta - z),$$

where η is the elevation of the free surface above that point. The quantity that we need is the horizontal pressure gradient $\partial p/\partial x = \rho g \, \partial\eta/\partial x$, and so the streamwise pressure gradient is entirely due to the slope of the free surface. The contribution is

$$-\int_A \frac{\partial p}{\partial x} \, \mathrm{d}V \approx -\rho \, \Delta x \, g \int_A \frac{\partial \eta}{\partial x} \, \mathrm{d}A \approx -\rho \, \Delta x \, gA \frac{\partial \eta}{\partial x}, \qquad (7.8)$$

where any variation with y has been ignored, as the surface elevation usually varies little across the channel, and so $\partial\eta/\partial x$ is constant over the section and has been able to be taken outside the integral, which is then simply evaluated.

c. **Resistance due to shear:** there is little that we can say that is exact about the shear forces.

We have already considered resistance in some detail, however, and in equation (2.17) we we have

$$\frac{\tau}{\rho} = \Lambda U^2 = \Lambda \left(\frac{Q\,(x,t)}{A\,(x,t)} \right)^2,$$

where we could use a value of Λ from the figure on page 31 or from the formulae given therein, or we could use our result for the Gauckler-Manning-Strickler formula, equation (2.20) where $\Lambda = 0.0223 \left(D/\left(A/P\right)\right)^{1/3}$. The value of τ is the mean around the solid boundary, so to obtain the force we multiply by the wetted perimeter P and length of the element Δx and instead of Q^2 we write $-Q\,|Q|$, to allow for possible negative Q in estuaries, as resistance always opposes the motion:

$$\text{Total horizontal shear force on control surface} = -\rho \, \Delta x \, \Lambda \frac{Q\,|Q|}{A^2} P. \qquad (7.9)$$

Collection of terms and discussion

Now all contributions from the hydraulic approximations to terms in equation (7.5) are collected, using equations (7.6), (7.7), (7.8), and (7.9), and bringing all derivatives to the left and others to the right, all divided by $\rho\Delta x$, gives the momentum equation:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta \frac{Q^2}{A}\right) + gA\frac{\partial \eta}{\partial x} = -\Lambda P \frac{Q\,|Q|}{A^2}. \qquad (7.10)$$

It is convenient to generalise the resistance term so as to be able to incorporate Gauckler-Manning-Strickler resistance as well as situations where a *Rating Curve* is known from river measurements,

giving a relationship between measured discharge, supposed steady and uniform, and local cross-sectional area, $Q_{\mathrm{r}}(A)$. We write the equation as

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta \frac{Q^2}{A}\right) + gA\frac{\partial \eta}{\partial x} = -\Omega Q\,|Q|\,, \qquad (7.11)$$

where the coefficient Ω, of dimensions L^{-3}, is a function of resistance coefficient, cross-sectional area, and wetted perimeter. The term can be written

$$\Omega = \begin{cases} gA\tilde{S}/Q_{\mathrm{r}}^2(A), & \text{in terms of rated discharge } Q_{\mathrm{r}}(A) \text{ and mean slope } \tilde{S}; \\ \Lambda P/A^2, & \text{Chézy-Weisbach, where } \Lambda = \lambda/8 = g/C^2; \\ n^2 g P^{4/3}/A^{7/3} = gP^{4/3}/k_{\mathrm{St}}^2 A^{7/3}, & \text{Gauckler-Manning.} \end{cases} \qquad (7.12)$$

These three have different aspects to them. The conventional form is to present the third one, in terms of Gauckler-Manning resistance. The awkwardness of the form is obvious. The other two forms might be preferable. The second, such that $\Omega = \Lambda PQ\,|Q|\,/A^2$, usually $\Omega = \Lambda PQ^2/A^2$, makes it clear that it is a resistance coefficient multiplied by the mean velocity squared to give stress, multiplied by the perimeter around which the stress acts. The first form, an innovation here, $\Omega = gA\tilde{S}/Q_{\mathrm{r}}^2(A)$, might be useful where (almost always!) the resistance coefficient is poorly-known but where there is a gauging station on the river, giving a measured relationship between discharge and surface elevation or area.

Example: Verify the use of the three resistance forms for steady uniform flow, on a uniform slope $\tilde{S} = S$.

In this case, the flow is steady so the first term in equation (7.11) is zero, and uniform so that the second is zero. The surface slope $\partial\eta/\partial x = -S$, and as Q is positive, $Q\,|Q| = Q^2$ and the momentum equation (7.11) gives $-gAS = -\Omega Q^2$, so that

$$Q = \sqrt{\frac{gAS}{\Omega}} = \begin{cases} \sqrt{\frac{gAS}{gAS}Q_{\mathrm{r}}^2} = Q_{\mathrm{r}}, & \text{in terms of rated discharge } Q_{\mathrm{r}}(A); \\ \sqrt{\frac{gA^3S}{\Lambda P}} = A\sqrt{\frac{g}{\Lambda}\frac{A}{P}S}, & \text{Chézy-Weisbach;} \\ \sqrt{\frac{gAS}{n^2 gP^{4/3}}A^{7/3}} = \frac{1}{n}A\left(\frac{A}{P}\right)^{2/3}\sqrt{S}, & \text{Gauckler-Manning-Strickler.} \end{cases} \qquad (7.13)$$

At this stage the non-trivial assumptions in the derivation are stated, roughly in decreasing order of importance (they are actually not very restrictive at all!):

1. Resistance to flow is modelled empirically. The Navier-Stokes equations are not being used.
2. All surface variation is sufficiently long and of small slope that the pressure throughout the flow is given by the hydrostatic pressure corresponding to the depth of water above each point.
3. Effects of curvature of the stream course are ignored.
4. In the momentum flux term the effects of both non-uniformity of velocity over a section and turbulent fluctuations are approximated by a momentum or Boussinesq coefficient.
5. Surface elevation η across the stream is constant.

Relating surface slope $\partial\eta/\partial x$ and $\partial A/\partial x$

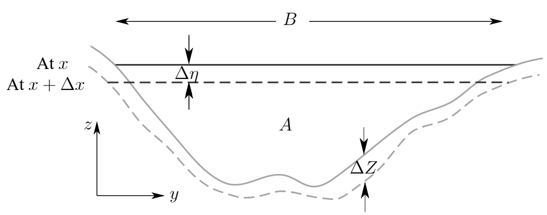


Figure 7.2: Two channel cross-sections separated by Δx

In the momentum equation (7.10) when expanded, the dependent variables are discharge Q and a mixture of derivatives of area $\partial A/\partial x$ and surface elevation $\partial\eta/\partial x$. We must relate the two, and now consider the bottom geometry in greater detail, although in practice the precise details of the bed are often not known. This will help us know when to make approximations.

The cross-section of a river in Figure 7.2 shows how ambiguous and possibly non-unique the concept of the "bottom" of the stream may be. In a distance Δx the surface elevation may change by an amount $\Delta\eta$ as shown, so that the contribution to the change in cross-section area ΔA is $B\times\Delta\eta$, where $\Delta\eta$ is usually negative as the surface drops downstream. The change in the bed is ΔZ, which in general varies across the section, with contribution to ΔA of $-\int_B \Delta Z\, dy$, the area between the solid and dotted lines on the figure corresponding to the bed at the two locations. The minus sign is because, if the bed drops away and ΔZ is negative, as usual, the contribution to area

increase is positive. Combining the two terms,

$$\Delta A = B\,\Delta\eta - \int\limits_B \Delta Z\,\mathrm{d}y \tag{7.14}$$

For the second contribution, the integral of the change in bed elevation across the stream, we introduce the symbol $\tilde{S}$ for the mean downstream bed slope across the section such that

$$\tilde{S} = -\frac{1}{B}\int\limits_B \frac{\partial Z}{\partial x}\,\mathrm{d}y, \tag{7.15}$$

where the negative sign has been introduced such that in the usual case when Z decreases with x, this mean downstream bed slope at a section is positive. In an important problem where bed details might be known, this can be evaluated. In the usual case where bed topography is poorly known, a reasonable local approximation or assumption is made. Using equations (7.14) and (7.15) we can write

$$\Delta A = B\,\Delta\eta + B\tilde{S}\,\Delta x,$$

where in a distance Δx the *mean* bed level across the channel then changes by $-\tilde{S}\times\Delta x$ under the water. In the rare case where the sides of the stream are vertical diverging or converging walls, an extra term would have to be included. Taking the usual calculus limit, we obtain

$$\frac{\partial A}{\partial x} = B\left(\frac{\partial\eta}{\partial x} + \tilde{S}\right), \tag{7.16}$$

which might have been able to have been written down without the mathematical details.

7.3 Forms of the governing equations

We use equation (7.16) to eliminate first $\partial\eta/\partial x$ and then $\partial A/\partial x$ to give two alternative forms of the momentum equation governing flows and long waves in waterways. In both cases, we restate the corresponding mass conservation equation, using (7.1) and (7.4), to give the pairs of equations:

Formulation 1 – Long wave equations in terms of area A and discharge Q

Eliminating $\partial\eta/\partial x$ gives the equations in terms of A and Q:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = i, \tag{7.17a}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right) + \frac{gA}{B}\frac{\partial A}{\partial x} = gA\tilde{S} - \Omega Q\,|Q|\,. \tag{7.17b}$$

Formulation 2 – Long wave equations in terms of stage η and discharge Q

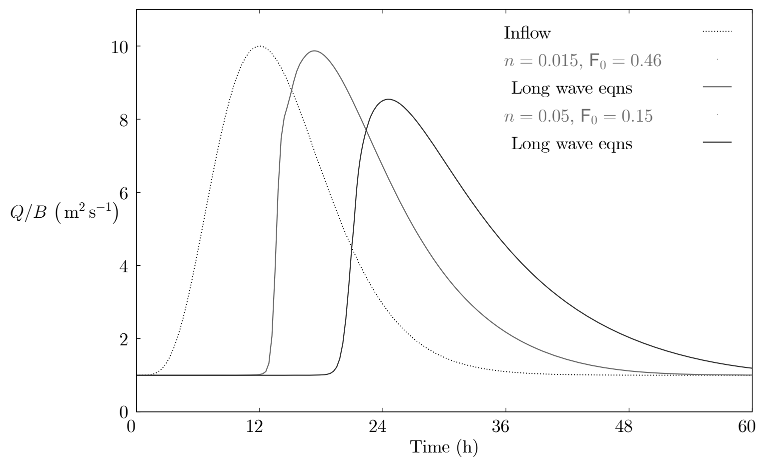
Now eliminating $\partial A/\partial x$, but retaining A in all coefficients, as it can be calculated in terms of η:

$$\frac{\partial\eta}{\partial t} + \frac{1}{B}\frac{\partial Q}{\partial x} = \frac{i}{B}, \tag{7.18a}$$

$$\frac{\partial Q}{\partial t} + 2\beta\frac{Q}{A}\frac{\partial Q}{\partial x} + \left(gA - \beta\frac{Q^2 B}{A^2}\right)\frac{\partial\eta}{\partial x} = \beta\frac{Q^2 B}{A^2}\tilde{S} - \Omega Q\,|Q|\,. \tag{7.18b}$$

These equations are the basis of computational hydraulics and flood routing. There is much commercial software written to solve them. They are actually quite simple in the form here!

7.4 Examples of flood propagation



As an example we consider an infinitely-wide (no side friction) channel with a channel slope $S = 0.0005$ and length 50 km. Two different boundary resistances were considered

- $n = 0.015$ for a smooth boundary (little rougher than smooth cement) to give a large Froude number. In this artificial canal case the wave has steepened, with little diffusion.
- $n = 0.05$ for a rough natural boundary, more likely in practice, with finite diffusion.

7.5 A practical recommendation to use the (A, Q) form of the equations

The equations have as parameters: inflow i, usually ignored; momentum coefficient β which is only approximately known; area A; breadth B; slope $\tilde{S}$, approximately known; and in the resistance term additionally the wetted perimeter P and resistance coefficient (Λ, n, or k_{St}), also approximately known. In the (η, Q) form, equations (7.18), it is necessary to know the geometrical functional relationships $A(x,\eta)$, $B(x,\eta)$ and $P(x,\eta)$, at all points in the stream. This requires a specification of the underwater geometry; it is possible that commercial software requires the input of assumed cross-sections. However, it is likely that the geometry is actually poorly known, and so the trouble of going to assumed forms of dependence is questionable.

One can and does assume approximate values of $\tilde{S}$ and resistance coefficient. The formulation of equations (7.17) in terms of A allows an approximate procedure that is commensurate with the accuracy of knowledge of other parts of the problem. As A is one of the dependent variables, ideally one also has the geometrical problem of obtaining the corresponding $B(x, A)$ and $P(x, A)$. For most streams B and P are not going to vary much with flow anyway. It would be appropriate to assume approximate, possibly constant, values of B and P for each computational point, possibly from aerial photography or simply assumed values, and then to make the wide stream approximation $P(x, A) \approx B(x, A) = B_0(x) \equiv$ Constant.

Surprisingly, no other geometric information is necessary. Hence, using (A, Q) one can perform model simulations with relatively little information required or included artificially.

7.6 Nature of the equations and solutions

The Telegraph equation as a model for long wave propagation

We recall the long wave equations in terms of area, equations (7.17):

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = i,$$
$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right) + \frac{gA}{B}\frac{\partial A}{\partial x} = gA\tilde{S}\left(1 - \frac{Q^2}{Q_{\mathrm{r}}^2(A)}\right),$$

where we have now written the resistance term in terms of the function of area $Q_{\mathrm{r}}(A)$ which is the Rating Curve relationship, or that given by, say, the Gauckler-Manning formula. For steady uniform flow such that all derivatives on the left are zero, the equation becomes $Q = Q_{\mathrm{r}}(A)$, which we require. We also recall the relationships (7.3) for the upstream volume:

$$\frac{\partial V}{\partial x} = A \quad \text{and} \quad \frac{\partial V}{\partial t} = \int^{x} i\,(x')\,\mathrm{d}x' - Q. \tag{7.19}$$

Substituting these into the mass conservation equation, as we did in §7.1.1, we find that it is identically satisfied (we expect volume to satisfy a volume conservation equation). The momentum conservation equation becomes

$$\frac{\partial^2 V}{\partial t^2} + 2\beta\frac{Q}{A}\frac{\partial^2 V}{\partial x \partial t} + \left(\beta\frac{Q^2}{A^2} - \frac{gA}{B(A)}\right)\frac{\partial^2 V}{\partial x^2} + gA\tilde{S}\left(1 - \left(-\frac{\partial V/\partial t}{Q_{\mathrm{r}}(A)}\right)^2\right) = 0$$

where symbols Q and A have been retained in coefficients of second derivatives.

- The momentum equation has become a second-order partial differential equation in terms of the single variable V.
- And it is unusable in this form with difficult terms with time derivatives. It is more useful in theoretical works and where approximations can be made, as we now do.
- We linearise the equation by considering relatively small disturbances about a uniform flow with area A_0 and discharge Q_0. Substituting the series

  $$V = A_0 x - Q_0 t + \varepsilon v,\ A = A_0 + \varepsilon v_x,\ Q = Q_0 - \varepsilon v_t,\ \text{and}\ Q_{\mathrm{r}}(A) = Q_0 + Q_0'\,\varepsilon v_x,$$

  where εv is a small quantity, a deviation of upstream volume from that of uniform flow, $v_t = \partial v/\partial t$, $v_x = \partial v/\partial x$, and $Q_0' = \mathrm{d}Q_{\mathrm{r}}/\mathrm{d}A|_0$.

Performing power series operations, the second order derivatives can simply be written down with constant coefficients. The gravity and resistance terms become, making use of the power series expansion to first order, $(1+\varepsilon a)^n = 1 + \varepsilon n a + \ldots$:

$$g\,(A_0 + \varepsilon v_x)\,S_0 \times \left(1 - \left(\frac{Q_0 - \varepsilon v_t}{Q_0 + Q_0'\,\varepsilon v_x}\right)^2\right) = gA_0S_0 \times \left(1 - \left(\frac{1 - \varepsilon v_t/Q_0}{1 + \varepsilon Q_0'/Q_0\,v_x}\right)^2\right)$$
$$= \varepsilon\frac{2gA_0S_0}{Q_0}\left(\frac{\partial v}{\partial t} + Q_0'\frac{\partial v}{\partial x}\right).$$

We obtain the linearised momentum equation as the *Telegraph equation*:

$$\sigma_0\left(\frac{\partial v}{\partial t} + c_0\frac{\partial v}{\partial x}\right) + \frac{\partial^2 v}{\partial t^2} + 2\beta U_0\frac{\partial^2 v}{\partial x \partial t} - \left(C_0^2 - \beta^2\,U_0^2\right)\frac{\partial^2 v}{\partial x^2} = 0. \tag{7.20}$$

- σ_0 – **Resistance parameter / inverse time scale**: this is actually an important channel parameter, determining the nature of wave behaviour and computational solution properties

  $$\sigma_0 = \frac{2gA_0S_0}{Q_0} = 2\frac{gS_0}{U_0} = \frac{\partial}{\partial Q}\left(gAS\frac{Q^2}{Q_{\mathrm{r}}^2}\right)\Bigg|_0$$

  It is the derivative with respect to Q of the resistance term in the momentum equation. We could argue by a rough electrical analogy that the resistance term in the momentum equation is equivalent to potential difference or voltage, while discharge Q is equivalent to current. As the derivative of voltage with respect to current gives electrical resistance, σ_0 can be thought of as a *resistance parameter* in our nonlinear case.
- c_0 – **Very long wave speed**: This will be shown to be the speed of very long period waves, which means for us the propagation speed of flood waves:

  $$c_0 = \frac{\mathrm{d}Q_{\mathrm{r}}}{\mathrm{d}A}\bigg|_0. \tag{7.21}$$

  Using the Gauckler-Manning equation $Q_{\mathrm{r}} = 1/n \times A^{5/3}/P^{2/3}\sqrt{S}$, for a wide stream, ignoring change of P with A, this gives $c_0 \approx \frac{5}{3}U_0$, so that a good estimate of the speed of propagation of a flood wave is to multiply the stream velocity by $5/3$. This wave speed is important, and will be studied more practically later.

- $U_0 = Q_0/A_0$ – **mean fluid velocity**: used for simplicity.
- C_0 – the **speed of not-so-long waves**:

$$C_0 = \sqrt{gA_0/B_0 + \left(\beta^2 - \beta\right) U_0^2}, \tag{7.22}$$

  In most textbooks this is written, not unreasonably, implicitly with $\beta = 1$ such that $C_0 = \sqrt{gA_0/B_0}$, which is usually said to be the "celerity" or "long wave speed" or "dynamic wave speed". Below it will be shown that it is actually the speed of waves only in the limit of shorter waves, but still long enough that the hydrostatic approximation holds. We call these "not-so-long" waves. They occur when waves are due to rapid gate movements. This velocity is less-important than is generally believed.

We now obtain some simple solutions to the Telegraph equation in two limits.

Very long waves – the longest flood waves

- For disturbances with a long period, such that $\partial^2/\partial t^2 \ll \sigma_0 \partial/\partial t$, "very long waves", the last three terms in the equation can be neglected, and it becomes the advection equation, the *Very Long Wave equation*

$$\frac{\partial v}{\partial t} + c_0 \frac{\partial v}{\partial x} = 0, \text{ Solution: } v = f_1\left(x - c_0 t\right), \tag{7.23}$$

  where $f_1\left(.\right)$ is an arbitrary function given by the upstream conditions. To show this consider a moving variable $X = x - c_0 t$, and $v = f_1(X)$. By the *chain rule* for partial differentiation,

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial f_1(X)}{\partial t} = \frac{df_1(X)}{dX}\frac{\partial X}{\partial t} = -c_0 \frac{df_1(X)}{dX}, \quad \text{and} \\ \frac{\partial v}{\partial x} &= \frac{\partial f_1(X)}{\partial x} = \frac{df_1(X)}{dX}\frac{\partial X}{\partial x} = 1 \times \frac{df_1(X)}{dX}, \end{aligned}$$

  and the equation is satisfied for *any* $f_1(X)$, whatever the upstream conditions determine.
- This solution is a wave propagating downstream at speed c_0 with no change or diffusion.
- The equation has been known as the "kinematic wave equation" and c_0 the "kinematic wave speed", because the approximation has previously been believed to be such that dynamic terms of order F^2 in the momentum equation have been neglected. Here we have shown that *the only approximation has been that the wave period is long*. A better name is the *Very Long Wave Equation*, VLWE, as the theory is a long wave theory (waves are long compared with depth) but here we are at the long end of the long wave theory.

Not-so-long waves – in the shorter limit of waves from the long wave equations

- In the other limit, for disturbances which are shorter, such that $\partial^2/\partial t^2 \gg \sigma_0 \partial/\partial t$, for which we use the term "not-so-long" waves, the Telegraph equation becomes

$$\frac{\partial^2 v}{\partial t^2} + 2\beta U_0 \frac{\partial^2 v}{\partial x \partial t} - (C_0^2 - \beta^2 U_0^2)\frac{\partial^2 v}{\partial x^2} = 0,$$

  which is a second-order wave equation with solutions

$$v = f_{21}\left(x - \left(\beta U_0 + C_0\right) t\right) + f_{22}\left(x - \left(\beta U_0 - C_0\right) t\right)$$

  where $f_{21}\left(.\right)$ and $f_{22}\left(.\right)$ are arbitrary functions determined by boundary conditions both upstream and downstream.
- In this case the solutions are waves propagating upstream and downstream at velocities of $\beta U_0 \pm C_0$, such that in the usual terminology C_0 is the "long wave speed", and the waves travel relative to an advection velocity βU_0, where the presence of β is slightly surprising.
- We have shown here that C_0 is the actually the speed of waves that are not so long, apparently paradoxically – they are long enough that the pressure distribution in the fluid is still hydrostatic, but they are short in terms of time scales given by the resistance characteristics.
- Such waves might be caused by relatively rapid changes such as the operation of gates.

Intermediate period waves

- In the general case, solutions of the long wave equations show wave propagation characteristics, velocity and rate of decay, that depend on the period of the waves, so that the waves are actually
  - diffusive – different period components decay at different rates, and
  - dispersive – different components travel at different speeds
- One can obtain solutions for the propagation behaviour in terms of wave period, but the operations are complicated, and they are not included here.
- The widespread belief, printed in all textbooks, is wrong, that all waves obeying the long wave equations travel at a speed $C \approx \sqrt{gA/B} \approx \sqrt{g \times \text{Depth}}$. The behaviour is very much more complicated. There is no such thing as "**the** long wave speed".

Solving the long wave equations numerically overcomes all such problems, but it is nice to know what physical processes are at work.

Propagation speed of flood waves: The nonlinear mathematical artefact, that flood waves do not travel at the fluid speed $U_0 = Q_\mathrm{r}(A_0)/A_0$, or at the misnamed "long wave speed" $\sqrt{gA/B}$, but at the very long wave speed $c_0 = \mathrm{d}Q_\mathrm{r}/\mathrm{d}A$, is not so obvious, physically! For the three different cases of the resistance formulations, equation (7.13) we find, where $P_0' = \mathrm{d}P/\mathrm{d}A|_0$:

$$c_0 = \begin{cases} \mathrm{d}Q_\mathrm{r}/\mathrm{d}A|_0\,, & \text{General expression} \\ \frac{3}{2}U_0\left(1-\frac{1}{3}A_0P_0'/P_0\right), & \text{Chézy-Weisbach} \\ \frac{5}{3}U_0\left(1-\frac{2}{5}A_0P_0'/P_0\right), & \text{Gauckler-Manning} \end{cases}. \tag{7.24}$$

This is actually one of the most important results for us: the speed at which a flood propagates is approximately $\frac{3}{2}$ or $\frac{5}{3}$ times the mean fluid speed in the river. As the latter follows from the generally-more-accurate Gauckler-Manning equation we might prefer that – but then we observe that the corrections due to perimeter changing will reduce that. We might prefer simply to use $\frac{3}{2}$.

Example: Estimate the effect of side resistance on flood wave speed for a river of bottom width $20\ \mathrm{m}$, side slopes 2:1 (H:V) and a depth of $2\ \mathrm{m}$. Using our relationships $B = W + 2mh$, $A = h(W+mh)$, and $P = W + 2\sqrt{1+m^2}h$ we have $\mathrm{d}P/\mathrm{d}h = 2\sqrt{1+m^2}$ and $\mathrm{d}A/\mathrm{d}h = B$

$$-\frac{2}{5}\frac{A_0}{P_0}\frac{\mathrm{d}P}{\mathrm{d}A}\bigg|_0 = -\frac{2}{5}\frac{A_0}{P_0}\frac{\mathrm{d}P/\mathrm{d}h}{\mathrm{d}A/dh}\bigg|_0 = -\frac{2}{5}\frac{A_0}{P_0}\frac{\mathrm{d}P/\mathrm{d}h}{B}\bigg|_0 = -\frac{2}{5}\frac{h\,(W+mh)}{W+2\sqrt{1+m^2}h}\frac{2\sqrt{1+m^2}}{W+2mh}$$

$$= -\frac{2}{5}\frac{2\,(10+2\times 2)}{10+2\times 2\sqrt{1+2^2}}\frac{2\sqrt{1+2^2}}{10+2\times 2\times 2} = -15\%$$

An unusually simple interpretation of the equations and wave behaviour

Consider the Telegraph equation (7.20), where we make the substitution $C_0^2 = gA_0/B_0 + \left(\beta^2-\beta\right)U_0^2$ from equation (7.22) to go back to physical quantities:

$$\sigma_0\left(\frac{\partial v}{\partial t} + c_0\frac{\partial v}{\partial x}\right) + \frac{\partial^2 v}{\partial t^2} + 2\beta U_0\frac{\partial^2 v}{\partial x\partial t} - (gA_0/B_0 - \beta U_0^2)\frac{\partial^2 v}{\partial x^2} = 0. \qquad \text{(Telegraph eqn)}$$

We now introduce the dimensionless co-ordinates (x_*, t_*) scaled according to the resistance parameter σ_0: $t_* = \sigma_0 t$ and $x_* = \sigma_0 x/c_0$, such that $\partial/\partial t = \sigma_0\partial/\partial t_*$ and $\partial/\partial x = \sigma_0/c_0 \times \partial/\partial t_*$.

Substituting into the Telegraph equation we obtain a surprisingly simple result:

$$\frac{\partial v}{\partial t_*} + \frac{\partial v}{\partial x_*} + \frac{\partial^2 v}{\partial t_*^2} + 2\frac{\beta}{\alpha}\frac{\partial^2 v}{\partial x_*\partial t_*} - \left(\frac{1}{\alpha^2\mathsf{F}^2} - \beta\right)\frac{\partial^2 v}{\partial x_*^2} = 0, \tag{7.25}$$

where $\alpha = c_0/U_0$, which is a channel-shape parameter, which we can almost consider a constant, with a value of $5/3$ for a wide channel; while momentum coefficient β is slightly greater than 1 to allow for non-uniformity of velocity over the cross-section and for turbulence; and $\mathsf{F}^2 = U_0^2/g(A_0/B_0)$ is the square of the Froude number. We have shown that, apart from near-constants α and β, the equation, and hence the nature of wave propagation in a river, contains a single parameter F^2! This is surprising.

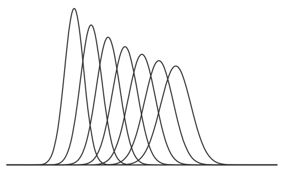
Advection-diffusion approximation

Now we consider a very useful approximation to describe wave propagation. We consider situations, such as for flood waves, where the changes in the system are relatively slow. This means that the second derivatives in the Telegraph equation are relatively small. The approximation in the limit of very long waves we have already obtained on page 90. Here, however, we observe that in the coefficient of $\partial^2 v/\partial x_*^2$ there is a term $1/\alpha^2\mathsf{F}^2$, and for most rivers this is a large number, as F^2 is small. Now including this term, we write equation (7.25) as, approximately,

$$\frac{\partial v}{\partial t_*} + \frac{\partial v}{\partial x_*} = \frac{1}{\alpha^2\mathsf{F}^2}\frac{\partial^2 v}{\partial x_*^2}, \tag{7.26}$$

where $1/\alpha^2\mathsf{F}^2$ is the diffusion coefficient, in this *Advection-Diffusion* equation. In our non-dimensional (x_*, t_*) variables, the advection velocity is 1. Now we consider the effects of the diffusion term with the second derivative.

In physics, the process of diffusion occurs because of a continuous process of random particle movements, where any irregularities in concentration of a substance are smoothed out. The significance of the equation is that any regions of curvature (where there is not a linear variation) will be smoothed out. For example, near a local maximum, $\partial^2 v/\partial x_*^2$ is negative, and this means that the contribution to $\partial v/\partial t_*$ there is negative, and the maximum is reduced and spread out. The reverse applies near a minimum.



Typical solutions of the equation are shown in the figure. It is a good simple approximate model of flood propagation. As a first estimate, we might assume that diffusion is unimportant, and that the flood peak might be the same downstream as it was upstream (as implied by the very long wave advection equation). For practical problems, however, the problem then arises as how to estimate the importance of diffusion.

To do this we write a solution of equation (7.26) as

$$v = v_0\exp\left(\mathrm{i}\left(\kappa x_* - \omega t_*\right)\right), \tag{7.27}$$

where v_0 is a coefficient, $\mathrm{i} = \sqrt{-1}$, and κ and ω are coefficients. We recall that $\exp(\mathrm{i}\theta) = \cos\theta + \mathrm{i}\sin\theta$, so it is a wave-like solution, which might be part of a Fourier series in a more general approach. Substituting into equation (7.26) and dividing through by v gives

$$-\mathrm{i}\omega + \mathrm{i}\kappa = \frac{-\kappa^2}{\alpha^2\mathsf{F}^2}. \tag{7.28}$$

If we consider κ to be a real number, corresponding to waves periodic in x_* this gives us a simple expression for ω which is complex, the real part giving us the period in time and the imaginary part giving us the rate of decay in time as the waves propagate. That is not so interesting for us, as we do not know the wave length. Instead, we have waves imposed on our river by an inflow hydrograph, in general as a Fourier series in time at the upstream end. We consider just one term of such a series, with a real frequency ω. We can write equation (7.28) as

$$1-\frac{\kappa}{\omega}+\mathrm{i}\mathsf{D}\left(\frac{\kappa}{\omega}\right)^2=0, \tag{7.29}$$

where all the propagation properties of the waves are contained in the now-complex κ/ω, for which we could solve this quadratic equation, the solution being a function of the dimensionless *Diffusion-frequency number*

$$\mathsf{D}=\frac{\omega}{\alpha^2\mathsf{F}^2}, \tag{7.30}$$

which expresses the importance of diffusion on the propagation of a wave of dimensionless frequency ω. The solution of equation (7.29) can be written down, but is non-trivial. Instead, here for simplicity we write κ/ω as a series in D and we find the solution

$$\frac{\kappa}{\omega}=1+\mathrm{i}\mathsf{D}+O\left(\mathsf{D}^2\right).$$

Thus we see that for small diffusion D solution (7.27) can be written

$$v\approx v_0\exp\left(\mathrm{i}\omega\left(x_*-t_*\right)\right)\times\exp\left(-\mathsf{D}x_*\right), \tag{7.31}$$

which is a wave propagating at speed 1 in our (x_*,t_*) co-ordinates, but decaying at a rate in space of $\exp\left(-\mathsf{D}x_*\right)$, showing us how shorter waves (larger ω and D) decay faster in space *and giving us a simple estimate of that decay*. Maybe we don't need simulation software and all that effort ...

Advection-diffusion equation in terms of physical quantities

Returning to physical variables, the *Advection-Diffusion Equation* is

$$\frac{\partial v}{\partial t}+c_0\frac{\partial v}{\partial x}=\frac{Q_0}{2B_0S_0}\frac{\partial^2 v}{\partial x^2}, \tag{7.32}$$

where the diffusion coefficient is $Q_0/2B_0S_0$. The terms on the left are our very long wave equation, showing the flood wave propagating with velocity c_0, but here modified by the second derivative or diffusion term, such that disturbances spread out in time. This equation, a good approximation for very long waves such as floods, shows us the nature of their movement. However in physical variables, the magnitude of the diffusion coefficient is not so obvious. It was more revealing to have it identified as $1/\alpha^2\mathsf{F}^2$in equation (7.26).

Note: We have written the equation in terms of v. Instead of v we can use Q or A as the dependent variable, as can be shown by differentiating the equation with respect of x or t and then using the definition of v.

A simplified momentum equation and a practical application

Considering the linear advection-diffusion equation as a longer-wave approximation to the linear Telegraph equation, we now make a similar approximation to the full momentum conservation equation, such that we neglect the time derivative and dynamic terms shown in pale blue here, making a *Very Long Wave* or *Slow Change* approximation, which is accurate for most river flows:

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right)+\frac{gA}{B}\frac{\partial A}{\partial x}=gA\tilde{S}\left(1-\frac{Q^2}{Q_\mathrm{r}^2}\right).$$

This can be re-arranged to give the momentum equation in the form of a simple expression for Q in terms of area A:

$$Q=\underbrace{Q_\mathrm{r}(A)}_{\text{Steady, uniform}}\times\sqrt{1-\frac{1}{B(A)S}\frac{\partial A}{\partial x}}. \tag{7.33}$$

It is interesting that even just using the first term, the rating expression $Q=Q_\mathrm{r}(A)$, where that could be given by the Gauckler-Manning or Chézy-Weisbach formulae, is already an approximation to the momentum equation. This approximate simple momentum equation (7.33) is surprisingly accurate, and can be used for other long wave propagation problems in streams. For example, it can be used to estimate the actual flow in a flood wave, giving a correction to the rated value $Q_\mathrm{r}(A)$ as the flood wave passes because the actual slope of the water surface is not that of the bed of the stream. The significance of the correction is that ahead of a flood wave, the cross-sectional area A is actually decreasing in the positive x direction, so that the contribution of $-\partial A/\partial x$ is positive, and the actual discharge is larger, corresponding to the greater downstream slope. Behind

the wave, the situation is reversed, and flow is less.

In practice, we do not know what the derivative $\partial A/\partial x$ is. However, at gauging stations we do have an accurate record of the variation of surface elevation with time, often being measured hourly. Using the very long wave equation for A as a first approximation we have $\partial A/\partial t+c_0\,\partial A/\partial x=0$. Eliminating the x derivative from equation (7.33), and using the simple relationship $\partial A/\partial t=B\,\partial\eta/\partial t$, we obtain the *Jones formula*:

$$Q=Q_\mathrm{r}(A)\sqrt{1+\frac{1}{c_0S}\frac{\partial\eta}{\partial t}}. \tag{7.34}$$

This requires using the very long wave speed c_0 but at a gauging station we know what $A(\eta)$ is, and so we could use the formulae (7.24) with $U_0=Q_\mathrm{r}(A)/A$, or simply $c_0\approx\frac{5}{3}Q_\mathrm{r}(A)/A$.

We have obtained a correction for the effects of variation with time (unsteadiness) on the discharge calculated and published from routine measurements of surface elevation ("stage").

Slow Change Routing Equation

We can use the simplified momentum equation to give us a simpler method for flood routing, the computation of the passage of such events. It is possible to substitute the formula (7.33) for Q into the mass conservation equation to give a single partial differential equation in area $A\,(x,t)$. However, the result is complicated, and inflow boundary conditions are often expressed in terms of a discharge hydrograph $Q_{\text{in}}(t)$, when the area A formulation is not so convenient.

A simpler and more general approach in terms of *Upstream Volume* can be developed. Substituting for $A = \partial V/\partial x$ and $Q = -\partial V/\partial t$ gives the single equation in the single dependent variable V:

$$\frac{\partial V}{\partial t} + Q_{\mathrm{r}}\,(\partial V/\partial x)\sqrt{1 - \frac{1}{S\,B(\partial V/\partial x)}\frac{\partial^2 V}{\partial x^2}} = 0\ , \tag{7.35}$$

where both breadth B and Q_{r} have been written as functions of area $A = \partial V/\partial x$. We call this the *Slow Change Routing Equation*. It is a single nonlinear equation in a single unknown. The only approximation relative to the long wave equations has been that the variation with time is slow. Boundary conditions involving discharge Q or stage η can be incorporated using equations (7.19) and the geometrical relationship between A and surface elevation η at a point. This is a fully nonlinear equation and gives accurate results. Its nature is rather clearer than the two long wave equations: the second derivative term looks familiar ... We can relate the formula to the advection-diffusion equation obtained previously by making a linear approximation. We expand

the square root term using the binomial theorem and obtain the equation

$$\frac{\partial V}{\partial t} + Q_{\mathrm{r}}\,(\partial V/\partial x) - \tfrac{1}{2}\frac{Q_{\mathrm{r}}\,(\partial V/\partial x)}{S\,B(\partial V/\partial x)}\frac{\partial^2 V}{\partial x^2} = 0\ , \tag{7.36}$$

and we now further linearise, as when we obtained the Telegraph equation. We consider a general unsteady flow superimposed on a steady uniform flow, of area A_0 and discharge $Q_0 = U_0 A_0$, such that we write $V = A_0 x - U_0 A_0 t + v\,(x,t)$, where $v\,(x,t)$ is the (relatively small) unsteady or non-uniform contribution. The substitution into equation (7.36) is simple – except for the advective term. We write

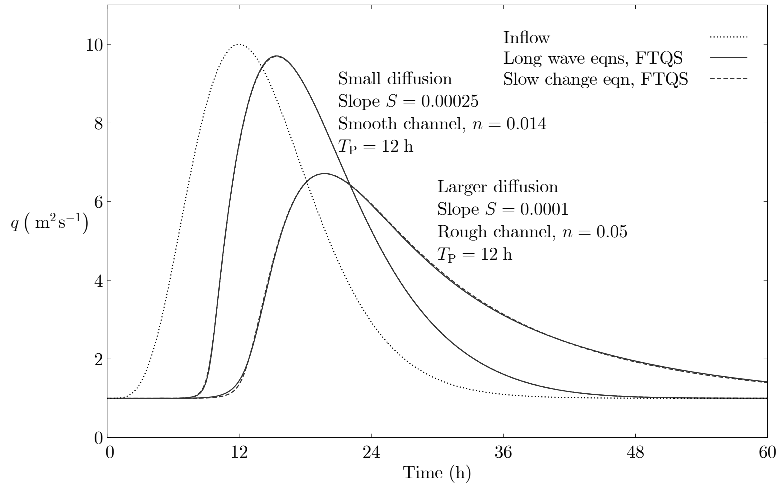
$$\begin{aligned} Q_{\mathrm{r}}\,(\partial V/\partial x) &= Q_{\mathrm{r}}\,(A_0 + \partial v/\partial x) \\ &= Q_{\mathrm{r}}\,(A_0) + \left.\frac{\mathrm{d}Q_{\mathrm{r}}}{\mathrm{d}A}\right|_0 \frac{\partial v}{\partial x} + \ldots, \end{aligned}$$

using the first terms of a Taylor series. However, $Q_{\mathrm{r}}\,(A_0) = Q_0$ and for $\mathrm{d}Q_{\mathrm{r}}/\mathrm{d}A$ already have the symbol c_0, the very long wave speed. The result is the advection-diffusion equation again:

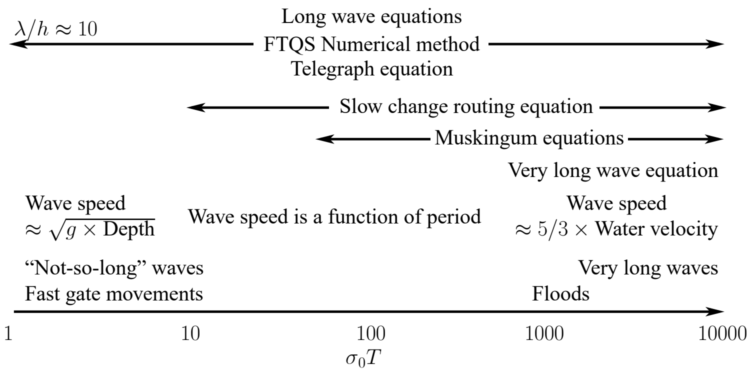
$$\frac{\partial v}{\partial t} + c_0\frac{\partial v}{\partial x} = \frac{Q_0}{2B_0 S}\frac{\partial^2 v}{\partial x^2}.$$

Two examples, showing smaller and greater diffusion effects

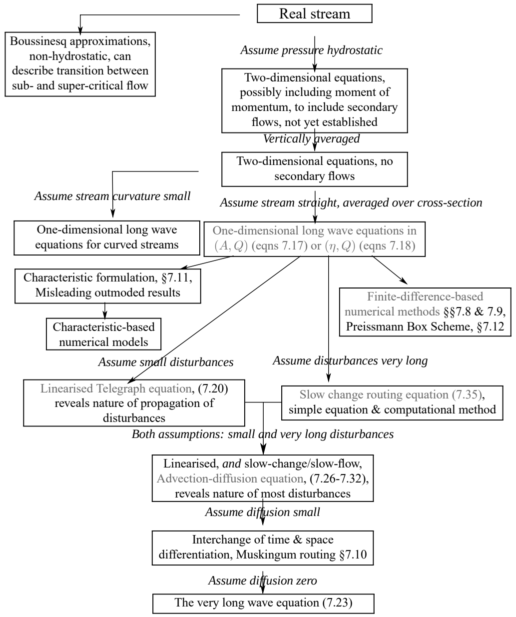
The figure shows results from two computations, for a $50\,\mathrm{km}$ length of river, first with a very smooth boundary and steeper slope so that diffusion is not so large, the second for a rougher boundary and smaller slope.



Summary of theories, names, and ranges of application



7.7 Hierarchy of one-dimensional open channel theories and approximations



7.8 Numerical solution of the long wave equations – FTQS scheme

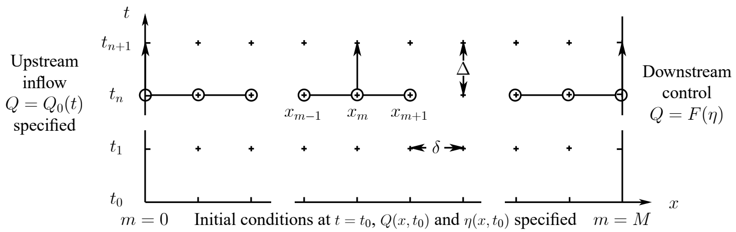


Figure 7.3: (x, t) axes showing computational grid, initial and boundary conditions, and three computational modules

We use a scheme where time derivatives are approximated using forward differences, and where x-derivatives are approximated using quadratic approximation, fitting a quadratic to three points, giving the **F**orward-**T**ime-**Q**uadratic-**S**pace scheme. The x-derivatives are

$$\left.\frac{\partial f}{\partial x}\right|_0 = \frac{-3f_0 + 4f_1 - f_2}{2\delta} + O(\delta)\ , \tag{7.37a}$$

$$\left.\frac{\partial f}{\partial x}\right|_m = \frac{f_{m+1} - f_{m-1}}{2\delta} + O(\delta)\,, \quad \text{for } m = 1, \ldots, M-1\ , \tag{7.37b}$$

$$\left.\frac{\partial f}{\partial x}\right|_M = \frac{f_{M-2} - 4f_{M-1} + 3f_M}{2\delta} + O(\delta)\ . \tag{7.37c}$$

Using the obvious forward difference expressions for the time derivatives, the scheme applied to the (A, Q) formulation, equations (7.17), becomes

$$\frac{A_{m,n+1} - A_{m,n}}{\Delta} = i - \left.\frac{\partial Q}{\partial x}\right|_{m,n}, \tag{7.38a}$$

$$\frac{Q_{m,n+1} - Q_{m,n}}{\Delta} = -\frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right) - \frac{gA}{B}\frac{\partial A}{\partial x} + gA\tilde{S} - \Omega Q\left|Q\right|\Bigg|_{m,n}. \tag{7.38b}$$

Both expressions are easily re-arranged to give explicit formulae for the terms in red, the values of A and Q at point m at the next time level $n+1$, each in terms of three values of A and three values of Q at the computational points at the previous time t_n, using one of the equations (7.37).

Liggett & Cunge (1975) claimed that the above scheme, the simplest and most obvious, was unconditionally unstable. This had some important implications, for it meant that the world was forced into using complicated schemes such as the Preissmann Box scheme (page 118), which form the basis of all commercial software. Their analysis is wrong (Fenton 2014), the scheme has a quite acceptable stability limitation, and it opens up the possibility for simpler computations of floods and flows in open channels. The Preissmann Box Scheme allows much larger time steps, but it is very complicated to apply.

7.9 Initial and boundary conditions

Initial conditions

Usually there is some initial flow in the channel which is constant if there is no inflow, $Q(x, t_0) = Q_0$. The next step is to determine the initial distribution of area A or surface elevation η. The conventional method is to solve the Gradually-varied flow equation, using the equations and methods described in §8, as well as the downstream boundary condition, which is about to be described. A simpler method is to use the unsteady equations and computation scheme that will be used later anyway – simply start with an approximate solution for $A(x, t_0)$ or $\eta(x, t_0)$ (a straight line?) and let the unsteady dynamics take over, allowing disturbances to propagate downstream and out of the computational domain until the solution is steady. Then, for example, the main computation can be started.

Boundary conditions

Upstream

It is usually the upstream boundary condition that drives the whole model, where a flood or wave enters, via the specification of the time variation of $Q = Q(x_0, t)$ at the boundary. The surface elevation there is obtained as part of the computations. A common model inflow hydrograph is:

$$Q(x_0, t) = Q_{\min} + (Q_{\max} - Q_{\min}) \left(\frac{t}{T_{\max}} e^{1-t/T_{\max}} \right)^5,$$

where the event starts at $t = 0$ with $Q_{\min}$ and has a maximum $Q_{\max}$ at $t = T_{\max}$.

We have two variables, however, Q and either A or η. To obtain this we just use the mass conservation FTQS expression (7.38a) to obtain the updated value of A or η at $m = 0$ at t_{n+1}. The equation of course applies up to and including the boundary point.

Downstream boundary - known stage-discharge relationship

- Where there is a downstream control structure such as a spillway, weir, gate, or flume, the stage-discharge relationship $Q(x_M, t) = F(\eta(x_M, t))$ must be known. For example, a weir might have a flow formula such as

$$Q = C_{\mathrm{d}} \sqrt{g} b \, (\eta - z_{\mathrm{c}})^{3/2}$$

  where $C_{\mathrm{d}} \approx 0.6$ is a weir discharge coefficient, b is crest length and z_{c} is elevation of the crest.
- We assume that the discharge relationship is not affected by unsteadiness and non-uniformity, which probably holds for relatively short control structures mentioned
- A potential difficulty – we have one equation too many: we have the FTQS finite difference formulae based on mass conservation for $\eta|A(x_M, t_{n+1})$ and momentum conservation for $Q(x_M, t_{n+1})$ *and* the relation between Q and $A|\eta$
- However, a sudden change in section where a typical spillway, weir, gate, or flume is placed actually violates a fundamental assumption of the long wave momentum equation, that variation in the channel is long. We can easily ignore that equation near such a sudden change
- Fortunately, the mass conservation equation, is still valid near a sudden change – it requires only the assumption that water surface is horizontal across the channel.
- The procedure is: obtain the updated value $A|\eta(x_M, t_{n+1})$ from the FTQS finite difference formula for the *mass conservation* equation (7.38a), using values of Q at x_{M-2}, x_{M-1}, x_M and t_n and then use the *stage-discharge relationship* to calculate $Q(x_M, t_{n+1}) = F(A|\eta(x_M, t_{n+1}))$

Open downstream boundary

- A common boundary is where the computational domain is artificially truncated at some point in the stream. This is sometimes called *Normal Depth* boundary and standard practice is that the computational domain be artificially extended and this boundary condition be used far enough downstream from the study area that it does not affect the results there.
- The lecturer prefers a different approach, and this is simply to treat the boundary as if it were just any other part of the river (which it is!) and to use both long wave equations to update both $A|\eta$ and Q there, calculating the necessary derivatives $\partial A|\eta/\partial x$ and $\partial Q/\partial x$ from *upstream* finite difference formulae and simply treating the end point as if it were an ordinary point in the stream and using *both* FTQS formulae for $A|\eta(x_M, t_{n+1})$ and $Q(x_M, t_{n+1})$ there, with the three-point leftwards approximations for the last point x_M in terms of values at x_{M-2}, x_{M-1}, and x_M.
- This works very well in practice.

7.10 Muskingum methods

The advection-diffusion equation, re-written with the symbol D_0 for the coefficient of diffusion, $D_0 = Q_0/2B_0S$, is

$$\frac{\partial v}{\partial t} + c_0 \frac{\partial v}{\partial x} - D_0 \frac{\partial^2 v}{\partial x^2} = 0. \qquad \text{(Advection-diffusion equation)}$$

It is a good simple approximate model of flood propagation – not as good as the fully nonlinear *slow change routing equation*. Both have, however, a finite stability criterion – strangely, the numerical simulation with the second derivative diffusion term makes the computation of both less stable! However, numerical solution is not a problem – one simply takes smaller steps until it works. In the last 40 years, however, there have been a large number of papers published using *Muskingum methods*, named after a river in the USA where such a method was first applied. They mimic the advection-diffusion equation, are supposed to be simple and plausibly seem so, and have been widely used. People have obtained the methods, sometimes from a simple reservoir routing approach, sometimes from the long wave equations, using long, complicated and arbitrary methods. The problem is to obtain a single finite difference equation in a single variable. (Using upstream volume V solves that problem rather better!). A typical Muskingum scheme is written, where c_m^n is the very long wave speed $dQ_{\mathrm{r}}/dA|_m^n$, and D_m^n is the coefficient of diffusivity $D_m^n = Q_m^n/2B_m^nS$, and so on:

$$
\begin{aligned}
&\left(-\delta-\Delta c_m^n+2D_m^n/c_m^n\right)Q_m^n+\left(-\delta+\Delta c_{m+1}^n-2D_{m+1}^n/c_{m+1}^n\right)Q_{m+1}^n\\
&\quad+\left(\delta-\Delta c_m^{n+1}-2D_m^{n+1}/c_m^{n+1}\right)Q_m^{n+1}+\left(\delta+\Delta c_{m+1}^{n+1}+2D_{m+1}^{n+1}/c_{m+1}^{n+1}\right)Q_{m+1}^{n+1}=0\,.
\end{aligned}
$$

This can be re-arranged to give an explicit expression for Q_{m+1}^{n+1}, the top right point shown in the blue computational module in Figure 7.4 in terms of the known two values of the discharge at time n, Q_m^n and Q_{m+1}^n and the known value at the previous space point Q_m^{n+1}.

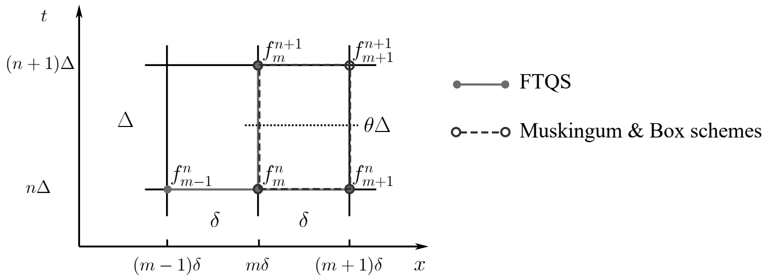


Figure 7.4: Computational stencils

However, if one performs a *Consistency Analysis*, and writes the point values Q_m^n *etc* as bi-dimensional Taylor series, one finds that the differential equation that the Muskingum formula

actually satisfies is

$$
\frac{\partial Q}{\partial t}+c_0\frac{\partial Q}{\partial x}+\frac{D_0}{c_0}\frac{\partial^2 Q}{\partial x\partial t}=0,
$$

and *not* the desired Advection-diffusion equation

$$
\frac{\partial Q}{\partial t}+c_0\frac{\partial Q}{\partial x}-D_0\frac{\partial^2 Q}{\partial x^2}=0.
$$

One can use the first two terms in the Muskingum equation to write $\partial Q/\partial t\approx -c_0\partial Q/\partial x$ and substitute this into the mixed derivative $\partial^2 Q/\partial x\partial t$ to give the advection-diffusion equation, but that is accurate only for small diffusion.

Muskingum methods work surprisingly well for small-diffusion problems, but in general, they solve the wrong equation, are numerically diffusive, and are to be avoided (Fenton 2019).

7.11 The method of characteristics

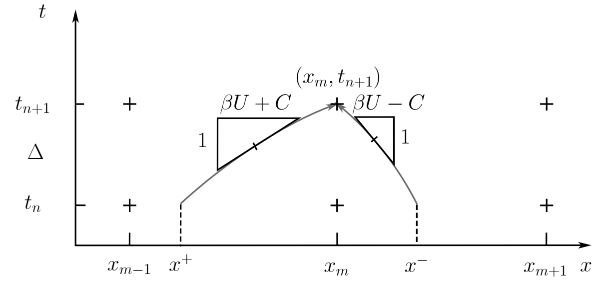


Figure 7.5: (x,t) axes, showing computational module with characteristics

This method is described in many books. The lecturer believes that it is something of an accident of history, and that the deductions that emerge from it are misleading and have caused several important misunderstandings about the nature of wave propagation in open channels.

Each of the pairs of long wave equations (7.17) and (7.18), which are partial differential equations, can be expressed as four ordinary differential equations. Two of the differential equations are for paths for $x(t)$, a path known as a *characteristic*:

$$
\frac{dx}{dt}=\beta U\pm C, \tag{7.39}
$$

where $U=Q/A$ is the mean fluid velocity in the waterway at that section and the velocity C is

$$
C=\sqrt{\frac{gA}{B}+U^2\left(\beta^2-\beta\right)},
$$

often incorrectly described as the “long wave speed”. It is, as equation (7.39) shows, the speed of the characteristics relative to the flowing water. The two contributions $\pm C$ correspond to downstream and upstream propagation of information. Two characteristics that meet at a point are shown on Figure 7.5. The “downstream” or “+” characteristic has a velocity at any point of $\beta U+C$. In the usual case where U is positive, both parts are positive and the term is large. As shown on the diagram, the “upstream” or “-” characteristic has a velocity $\beta U-C$, which is usually negative and smaller in magnitude than the other. Not surprisingly, upstream-propagating disturbances travel more slowly. The characteristics are curved, as all quantities determining them are not constant, but functions of the variable A, B, and Q.

The other two differential equations for η and Q can be established from the long wave equations:

$$
B\left(-\beta\frac{Q}{A}\pm C\right)\frac{d\eta}{dt}+\frac{dQ}{dt}=\beta\frac{Q^2B}{A^2}\tilde{S}-\Lambda P\frac{Q\,|Q|}{A^2}, \tag{7.40}
$$

On each of the two characteristics given by the two alternatives of equation (7.39), each of these two equations holds, taking the corresponding plus or minus signs in each case. To advance the solution numerically means that the four differential equations (7.39) and (7.40) have to be solved over time, usually using a finite time step Δ. Figure 7.5 shows the nature of the process on a plot of x against t.

The usual computational problem is, for a time $t_{n+1} = t_n + \Delta$, and for each of the discrete points x_m, to determine the values of x^+ and x^- at which the characteristics cross the previous time level t_n. From the information about η and Q at each of the computational points at that previous time level, the corresponding values of η^+, η^-, Q^+, and Q^- are calculated and then used as initial values in the two differential equations (7.40) which are then solved numerically to give the updated values $\eta\left(x_m, t_{n+1}\right)$ and $Q\left(x_m, t_{n+1}\right)$, and so on for all the points at t_{n+1}.

In textbooks and research papers, characteristics seem wrongly to be believed to have an almost supernatural property that the partial differential equations do not. An advantage of characteristics has been believed to be that numerical schemes are relatively stable. The lecturer is not convinced that they are any more stable then finite difference approximations to the original partial differential equations, but this remains to be proved conclusively.

In fact, the use of characteristics has led to a widespread misconception in hydraulics where C is understood to be the speed of propagation of all waves. It is not – it is the speed of *characteristics*. If surface elevation were constant on a characteristic there would be some justification in using the term ”wave speed” for the quantity C, as disturbances travelling at that speed could be observed. However as equation (7.40) holds, in general neither η (surface elevation – the quantity that we see), nor Q, is constant on the characteristics and one does not have observable disturbances, something that we would call a wave, travelling at C relative to the water. While C may be the speed of propagation of information in the waterway relative to the water, it cannot properly be termed the wave speed as it would usually be understood. In this course we have already examined at length the real nature of the propagation speed of waves.

7.12 Implicit methods – the Preissmann Box scheme

The most popular commercial numerical method for solving the long wave equations in time are Implicit Box (Preissmann) models, where the derivatives are replaced by finite-difference equivalents based on the rectangular blue module in Figure 7.4 on page 113:

$$\begin{aligned}
\frac{\partial f}{\partial x}(m,n) &\approx \frac{1}{\delta}\left[\theta\left(f_{m+1}^{n+1}-f_m^{n+1}\right)+(1-\theta)\left(f_{m+1}^{n}-f_m^{n}\right)\right], \\
\frac{\partial f}{\partial t}(m,n) &\approx \frac{1}{2\Delta}\left[\left(f_{m+1}^{n+1}-f_{m+1}^{n}\right)+\left(f_m^{n+1}-f_m^{n}\right)\right], \\
\bar{f}(m,n) &\approx \frac{1}{2}\left[\theta\left(f_{m+1}^{n+1}+f_m^{n+1}\right)+(1-\theta)\left(f_{m+1}^{n}+f_m^{n}\right)\right],
\end{aligned}$$

where θ is a coefficient that determines how much weight is attached to values at time $n+1$ (unknown, shown red) and how much to those at n (known, shown blue). Now, in the long wave equations (7.17) or (7.18) we use these expressions for all derivatives and also the averaged quantities $\bar{f}$ for those that occur algebraically. Considering all the modules at a certain time level, we have a set of $2M$ simultaneous complicated nonlinear algebraic equations in the values of Q and η at all points along the channel. The method is very complicated, but it is robust and stable, and large time steps can be taken. It is neutrally stable if $\theta = \frac{1}{2}$. In practice, one uses a larger value, such as $\theta = 0.6$, and the scheme is stable because it is computationally-diffusive. Several well-known commercial programs are available. For human purposes, it is simpler and better to use an explicit finite difference FTQS scheme.

7.13 Results

Evolution of flood wave – large diffusion case

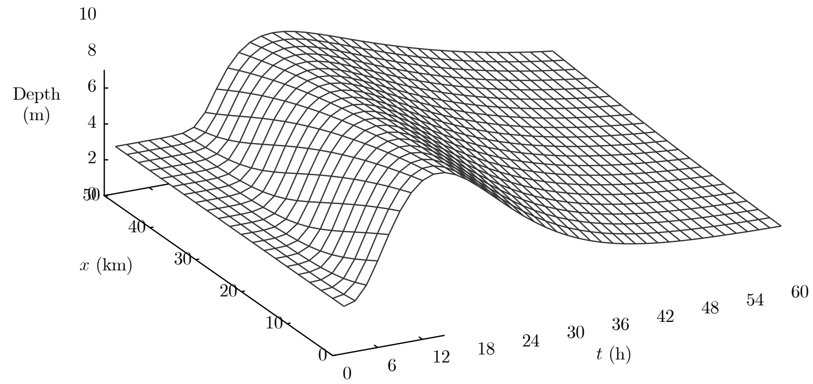
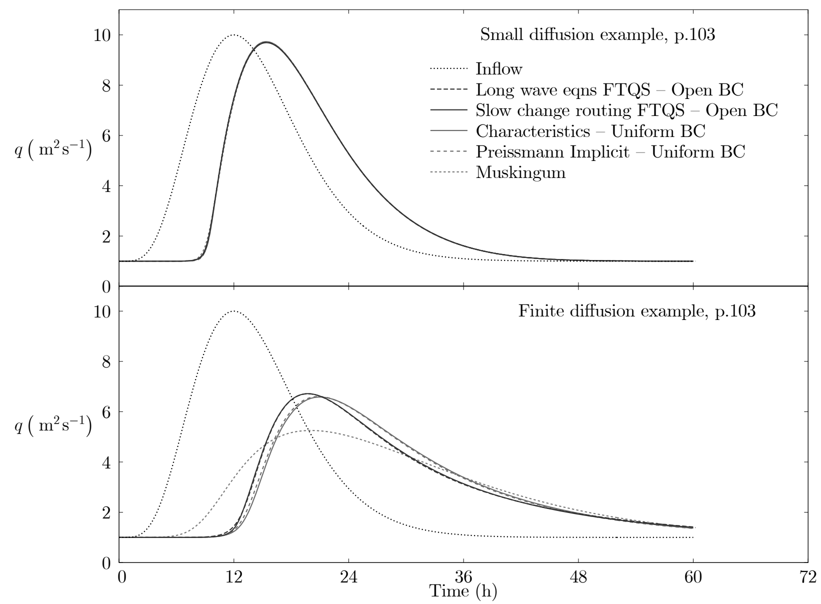


Figure 7.6:

Comparison of different methods



Conclusions

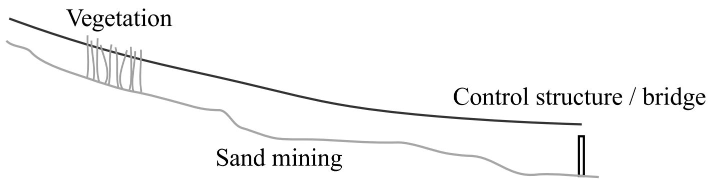
Small diffusion case:

- All methods performed well. Muskingum was accurate; the incorrect uniform flow downstream boundary condition did not matter (all motion is like simple advection – there is little diffusion for downstream effects to be felt upstream).

Finite diffusion case:

- Muskingum showed far too much diffusion when that was important. Such methods have been known to be problematical for small slopes, but nobody has called them out.
- The uniform flow downstream boundary condition: the Preissmann Implicit Box scheme and the Method of Characteristics agreed quite well with each other, but there were finite differences with those of the open downstream boundary condition.
- Both the FTQS finite difference schemes (solving the long wave equations and the slow change routing equation) agreed closely with each other using the more correct open boundary condition. They are the simplest methods and the best. One has to take relatively small time steps ($30\,\mathrm{s}$ used in the examples, compared with $900\,\mathrm{s}$ for the implicit and Muskingum methods), but their simplicity means that computational time is short.
- One can devise an example with a more rapidly-rising flood wave where the slow change routing equation no longer agrees so well. It is simple, and is in terms of a single variable, so that we used it to show the nature of approximations. *In general, however, solving the long wave equations themselves using our explicit FTQS scheme is the best of all.*

8. Steady flow



- A common task in river engineering is to calculate the free surface elevation along a steadily flowing stream.
- Simply the solution of a first-order differential equation – often obscured in writings.
- Flow is usually sub-critical, so the control / boundary condition is at the downstream end and one computes upstream.
- Alternative approach suggested here, using cross-sectional area as the dependent variable, requiring little knowledge of the details of the underwater topography.
- Traditional textbook methods are unsatisfactory: the "Standard Step" method is unnecessarily complicated and the "Direct Step" method is incorrect.
- Application of simple explicit numerical methods is described.
- If A is not used, for non-prismatic streams all methods require much data. Often that is not available. An approximate linearised model of flow in a river is made. This gives us insight into the nature of the problem, as well as simple approximate answers.

8.1 The gradually-varied flow equation (GVFE)

Use of area A and application to streams of unknown bathymetry

For steady flow where Q is constant so that $\partial Q/\partial t$ and $\partial Q/\partial x$ are zero, the long wave momentum equation (7.17b) on page 83 in terms of cross-sectional area A, gives one version of the GVFE in terms of area A:

$$\frac{\mathrm{d}A}{\mathrm{d}x} = B\frac{\tilde{S} - Q^2/K^2}{1 - \beta \mathsf{F}^2} = B\frac{\tilde{S} - n^2Q^2P^{4/3}/A^{10/3}}{1 - \beta Q^2 B/gA^3}\,. \tag{8.1}$$

In the resistance term we are using the conveyance K, which is a function of section properties and the Manning|Strickler coefficient $n|k_{\mathrm{St}}$

$$K = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}} = k_{\mathrm{St}}\frac{A^{5/3}}{P^{2/3}}, \tag{8.2}$$

such that for uniform flow, $\tilde{S} = S =$ constant, $Q_{\mathrm{r}} = K\sqrt{S}$.

The ordinary differential equation (8.1) is valid also for non-prismatic channels. The mean bed slope at a section $\tilde{S}$, can be variable but is, usually poorly known and is often just estimated, like the other parameters of the problem; β might be something like 1.1. The coefficient $n|k_{\mathrm{St}}$ is also often poorly known.

In the differential equation there are strongly-varying functions of the dependent variable itself, A^3 and possibly $A^{10/3}$, plus the usually slowly-varying functions $B(A)$ and $P(A)$. This suggests that using the GVFE in terms of A has an important advantage: *one needs few details of the under-water topography*. It is not necessary to know the precise details of the underwater bathymetry other than those weakly-varying functions $B(A)$ and $P(A)$. The obvious approximation could be made that they are constant and equal; river width often does not vary much.

To start numerical solution, one would need to know the area at a control where surface elevation might be known. The solution in terms of area might be enough, to give an idea of how far upstream the effects of a structure or channel changes extend. It is surprising that we can do so much with so little information. However, if one needed a value of surface elevation η at a certain value of x, one would then need cross-sectional details there to go from the computed A to η.

Customary use of a quantity h called the "water depth"

The long wave momentum equation (7.18b) in terms of surface elevation η, for Q constant so that $\partial Q/\partial t$ and $\partial Q/\partial x$ are zero gives another version of the GVFE:

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{\tilde{S}\beta\mathsf{F}^2 - Q^2/K^2}{1 - \beta\mathsf{F}^2}\ \left(\approx -Q^2/K^2 \text{ for } \mathsf{F}^2 \text{ small, the common case}\right)$$

The tradition is not to use η, but instead a depth-like quantity $h = \eta - Z_0$, where Z_0 is the elevation

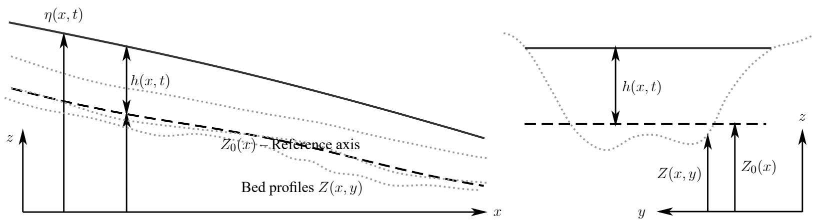


Figure 8.1: Complicated reality

of a longitudinal axis, almost always the supposed bed of the channel. The GVFE becomes

$$\frac{\mathrm{d}h}{\mathrm{d}x}=\frac{S_0+\beta\left(\tilde{S}-S_0\right)\mathsf{F}^2-Q^2/K^2}{1-\beta\mathsf{F}^2},$$

where $S_0=-\mathrm{d}Z_0/\mathrm{d}x$, the slope of the reference axis, positive downwards. We almost never know the details of $\tilde{S}$ so here we assume that $\tilde{S}=S_0$, which we now write as S, giving

$$\frac{\mathrm{d}h}{\mathrm{d}x}=\frac{S-Q^2/K^2}{1-\beta\mathsf{F}^2}$$

where in general both K and F are functions of both x and h, while in a prismatic channel, functions just of h.

Because of our use of h, we pretend that we know the bed in great detail, or, that our channel looks like this:

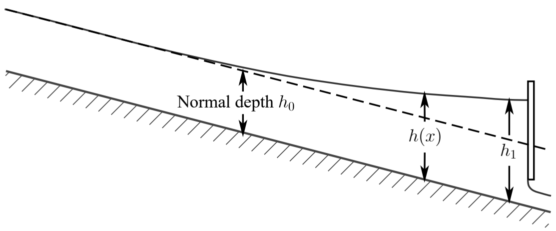


Figure 8.2: Our simpler model

This shows a typical subcritical flow retarded by a structure, showing the free surface disturbance decaying upstream, and if the channel is prismatic, to constant normal depth.

8.2 Traditional textbook methods – some old, complicated, and wrong

The "Standard" step method

The almost trivial energy derivation, ignoring non-prismatic effects, is that the rate of change of total head H is given by the empirical expression for the energy gradient

$$\frac{\mathrm{d}H}{\mathrm{d}x}=-Q^2/K^2\left(x,h\right)\quad\text{where}\quad H=Z_0(x)+h+\alpha\frac{Q^2}{2gA^2(x,h)}.$$

The computational approximation scheme is

$$\frac{H_{i+1}\left(h_{i+1}\right)-H_i\left(h_i\right)}{x_{i+1}-x_i}=-\tfrac{1}{2}Q^2\left(\frac{1}{K^2\left(x_i,h_i\right)}+\frac{1}{K^2\left(x_{i+1},h_{i+1}\right)}\right)$$

J. Fenton, Australia 1966;
H. Honsowitz, Austria, 1970?
solving transcendental equations

- The method advocated by Chow (1959) in a pre-computer era and still suggested by textbooks.
- $H(h)$ and $K(x,h)$ are both complicated geometrical functions of h, the unknown h_{i+1} is deep inside left and right sides.
- Requires numerical solution of a transcendental equation at each time step.

The "Direct" step method – distance calculated from depth

- Applied by taking steps in the water depth and calculating the corresponding step in x.
- It has some advantages: iterative methods are not necessary ("Direct").
- Practical disadvantages are:
  - It is applicable only to prismatic sections
  - Results are not obtained at specified points in x
  - As uniform flow is approached the steps become infinitely large
  - AND, it is wrong, as we now show

Consider the "specific head", the head relative to the local channel bottom, denoted here by H_0:

$$H_0(h)=H(h)-Z=h+\alpha\frac{Q^2}{2gA^2(h)}.$$

The differential equation becomes, after inverting each side

$$\frac{\mathrm{d}x}{\mathrm{d}H_0(h)}=\frac{1}{S-Q^2/K^2}.$$

A mistake and a correction

- The differential equation is now approximated, the left side by a finite difference expression $(x_i - x_{i+1}) / (H_{0,i} - H_{0,i+1})$.
- For the right side the numerical method as set out in textbooks is to take the mean of just the *denominator* at beginning and end points, and so to write

$$x_{i+1} = x_i + \frac{H_{0,i+1} - H_{0,i}}{\frac{1}{2}\left(S_i - Q^2/K_i^2 + S_{i+1} - Q^2/K_{i+1}^2\right)}$$

where the red shows the quantity that is a supposed mean value.

- While this is a plausible approximation, it is not mathematically consistent. What should be done is to use the mean value at beginning and end points of the *whole* right side of the differential equation, to give a trapezoidal approximation of the right side, which leads to

$$x_{i+1} = x_i + (H_{0,i+1} - H_{0,i})\,\frac{1}{2}\left(\frac{1}{S_i - Q^2/K_i^2} + \frac{1}{S_{i+1} - Q^2/K_{i+1}^2}\right).$$

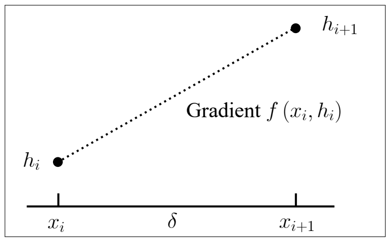
8.3 Standard simple numerical methods for differential equations

We write the differential equation as

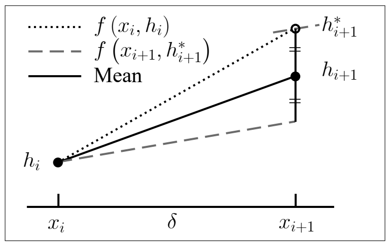
$$\frac{\mathrm{d}h}{\mathrm{d}x} = f(x,h) = \frac{S(x) - Q^2/K^2(x,h)}{1 - \beta \mathsf{F}^2(x,h)}$$

The two simplest numerical methods are:

Euler



Heun



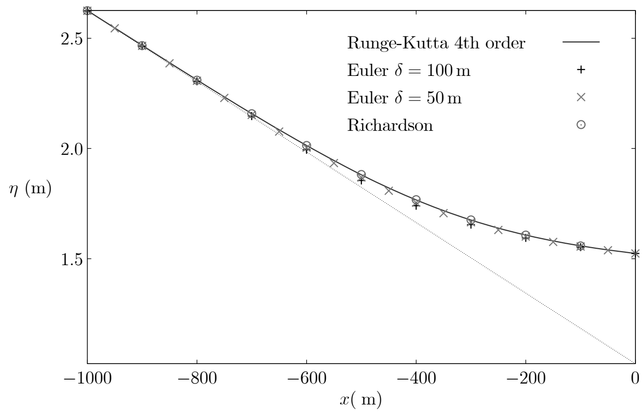
$$h_{i+1} \approx h_i + \delta\, f(x_i, h_i) + O\left(\delta^2\right)$$

$$h^*_{i+1} \approx h_i + \delta\, f(x_i, h_i),$$
$$h_{i+1} \approx h_i + \frac{\delta}{2}\left(f(x_i, h_i) + f\left(x_{i+1}, h^*_{i+1}\right)\right) + O\left(\delta^3\right)$$

- Euler's method is the simplest but least accurate – yet it might be appropriate for open channel problems where quantities may only be known approximately
- One can use simple modifications such as Heun's method to gain better accuracy, or use Richardson extrapolation, or even more simply, just take smaller steps δ
- For greater accuracy one can use the Trapezoidal method, simply repeating the second Heun step several times, setting $h^*_{i+1} = h_{i+1}$ each time
- Often these two methods are not presented in hydraulics textbooks as alternatives, yet they are simple and flexible, and reveal the nature of what we are doing
- The step δ can be varied at will, to suit possible irregularly spaced cross-sectional data
- In many situations, where $\mathsf{F}^2 \ll 1$, we can ignore the $\beta\mathsf{F}^2$ term in the denominators, giving a notationally simpler scheme

Comparison of schemes

Example 5 A flow of $11.33\ \mathrm{m^3\,s^{-1}}$ passes down a trapezoidal channel of gradient $S = 0.0016$, bed width $6.10\ \mathrm{m}$ and channel side slopes $H : V = 2$, $\alpha = \beta = 1.1$, and $k_{\mathrm{St}} = 40$. At $x = 0$ the flow is backed up to a depth of $1.524\ \mathrm{m}$. Compute the backwater curve for $1000\ \mathrm{m}$ in 10 steps and then 20, then perform Richardson extrapolation for a more accurate estimate.



Convergence of numerical schemes

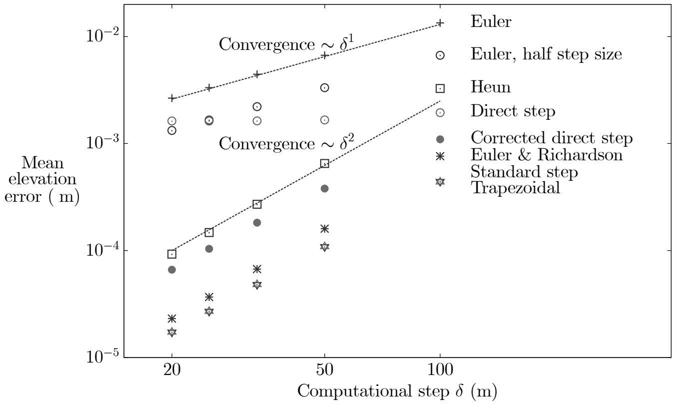


Figure 8.3: Comparison of accuracy - logarithmic scales

- Using Euler, then applying Richardson extrapolation, gave the third most accurate of all the methods, more than enough for practical purposes
- The most accurate were the Standard step method and the Trapezoidal method
- There *is* something wrong with the conventional Direct step method as we have suggested, while the corrected scheme is highly accurate

8.4 A simple[3] model of steady flow in a river

- Often the precise details of a stream are not known, and it is quite legitimate to make approximations
- These might give us more insight and understanding of the problem
- Now a model is made where the GVFE is linearised and a general solution obtained
- Simple deductions as to the length of backwater effects can be made
- One can calculate an approximate solution for a whole stream if the variation in the resistance coefficient and geometry are known or can be estimated
- There is more of a balance between what we know (usually little) and the (un)sophistication of the model

The GVFE is

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S - Q^2/K^2(x,h)}{1 - \beta \mathsf{F}^2(x,h)}$$

We linearise the problem (similar to obtaining the Telegraph equation) and consider small perturbations about an underlying uniform flow of slope S_0 and depth h_0, such that we write

$$h = h_0 + \varepsilon h_1(x) + \ldots,$$

where ε is a small quantity expressing the magnitude of deviations from uniform.

[3] Lecturer's little joke: we do some non-simple mathematics, but obtain a *very* simple result, equation (8.5) plus a formula for all streams.

Similarly we also let the possible non-constant slope be

$$S = S_0 + \varepsilon S_1(x) + \ldots.$$

In a real stream varying along its length, both K and F are functions of x and h. We write the series:

$$K = K_0 + \varepsilon K_1(x) + \varepsilon h_1(x) \left.\frac{\partial K}{\partial h}\right|_0 + O\left(\varepsilon^2\right) ,$$

where K_1 is a change caused by a change in the channel properties in x, whether the resistance coefficient or the cross-section, and $K_{h0} = \partial K/\partial h|_0$ expresses the change of conveyance with water depth. We also write

$$\mathsf{F}^2 = \mathsf{F}_0^2 + O\left(\varepsilon\right) + \ldots,$$

in which we will find that terms in ε are not necessary.

Multiplying through by $1 - \beta \mathsf{F}^2$, setting dh_0/dx to zero for uniform flow and neglecting terms in ε^2:

$$\varepsilon\left(1 - \beta \mathsf{F}_0^2\right)\frac{\mathrm{d}h_1(x)}{\mathrm{d}x} = S_0 + \varepsilon S_1(x) - \frac{Q^2}{K_0^2}\left(1 - 2\varepsilon\frac{K_1(x)}{K_0} - 2\varepsilon h_1(x)\frac{\partial K/\partial h|_0}{K_0}\right) .$$

At zeroeth order ε^0 we obtain

$$S_0 - Q^2/K_0^2 ,$$

an expression of whichever flow formula is being used, and is identically satisfied.

At ε^1, we obtain the linear differential equation

$$\frac{\mathrm{d}h_1}{\mathrm{d}x} - \gamma h_1 = \phi(x)$$

where γ is a constant:

$$\gamma = \frac{S_0}{1 - \beta \mathsf{F}_0^2} \times \begin{cases} 2\dfrac{\partial K/\partial h|_0}{K_0}, & \text{General expression;} \\ 3\dfrac{B_0}{A_0} - \dfrac{\mathrm{d}P/\mathrm{d}h|_0}{P_0}, & \text{Chézy-Weisbach;} \\ \frac{10}{3}\dfrac{B_0}{A_0} - \frac{4}{3}\dfrac{\mathrm{d}P/\mathrm{d}h|_0}{P_0}, & \text{Gauckler-Manning;} \end{cases} \tag{8.3}$$

and the forcing term on the right is

$$\phi(x) = \frac{S_0}{1 - \beta \mathsf{F}_0^2}\left(\frac{S_1(x)}{S_0} + \frac{2K_1(x)}{K_0}\right) , \tag{8.4}$$

showing the effects of fractional changes in slope and conveyance K.

Solving the differential equation

The differential equation is in *integrating factor* form, and can be solved by multiplying both sides by $e^{-\gamma x}$ and writing the result

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\gamma x} h_1\right) = e^{-\gamma x}\phi\left(x\right),$$

which can be integrated to give

$$h_1 = e^{\gamma x} \left(\int^x e^{-\gamma x'} \phi\left(x'\right)\, dx' + \text{Constant} \right),$$

where x' is a dummy variable. Returning to physical variables, $h = h_0 + \varepsilon h_1$ gives the solution

$$h = h_0 + He^{\gamma x} + \int^x e^{\gamma(x-x')} \phi\left(x'\right)\, dx'$$

The part of the solution $He^{\gamma x}$ is that obtained by Samuels (1989), giving the solution for backwater level in a uniform channel by evaluating the constant of integration using a downstream boundary condition $h = H$ at $x = 0$. The solution shows how the surface decays upstream at a rate $e^{\gamma x}$, as x becomes increasingly negative, because γ is positive,

- For a wide channel, the terms in dP/dh in the formulae for γ are unimportant (and are often not well known), so that $A_0/B_0 \approx h_0$, the channel depth, and for small Froude number this gives

$$\gamma \approx \frac{10}{3}\frac{S_0}{h_0} \approx 3\frac{S_0}{h_0}, \tag{8.5}$$

  showing that the rate of exponential decay is small for gently sloping and deep streams and greatest for steep and shallow ones.
- Consider the position $x_{1/2}$ upstream for the effect of a blockage to diminish by a factor of $1/2$.

Then $\exp\left(\gamma x_{1/2}\right) = 1/2$, or

$$x_{1/2} = \frac{\ln 1/2}{\gamma} \approx \frac{\ln 1/2}{3}\frac{h_0}{S_0} \approx -0.2\frac{h_0}{S_0}$$

So for a gently-sloping river $S_0 = 10^{-4}$ and $2\ \mathrm{m}$ deep, the effect of any backwater decreases by $1/2$ in a distance of $4\ \mathrm{km}$. To diminish to $1/16$, say, the distance is $16\ \mathrm{km}$. For a steeper river, say $S_0 = 0.0016$ from the example simulated above, where $h_0 \approx 1\ \mathrm{m}$, the "half-length" is about $150\ \mathrm{m}$. This is roughly in agreement with the computed results in Example 5 above.

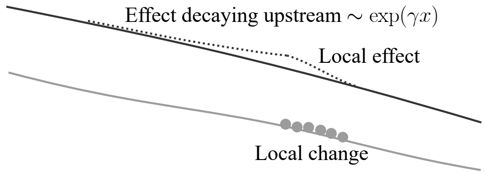
- If the approximate exponential decay solution were shown on that figure, it would not agree closely with the computed results, because the checked-up disturbance is as large as 50% of the depth, when the linear solution is not all that accurate. The beauty of Samuels' result is in its ability to give a quick estimate and an appreciation of the quantities that affect the length of backwater.

General solution for channel

Here we neglect any boundary conditions and consider just the solution due to the forcing function ϕ due to changes in the channel:

$$h = h_0 - \int_x^\infty e^{\gamma(x-x')} \phi\left(x'\right)\, dx' \tag{8.6}$$

This is a simple result: at any point x in subcritical flow, any disturbance is due to the integrated effects of the disturbance function ϕ for all downstream points, from x to ∞, weighted according to the exponential decay function.



Example 6 The effect on a river of a finite length of greater resistance

Consider, as an example, a case where over a finite length L of river, the carrying capacity is reduced by the conveyance K decreasing by a relative amount $K_1/K_0 = -\delta$, such as by local deposition of material, between $x = 0$ and $x = L$, and constant in that interval. Assume F_0^2 negligible and the river wide.

The forcing function form equation (8.4) is:

$$\phi(x) = \begin{cases} 0, & \text{if } x \leqslant 0; \\ -S_0\delta, & \text{if } 0 \leqslant x \leqslant L; \\ 0, & \text{if } x \geqslant L. \end{cases}$$

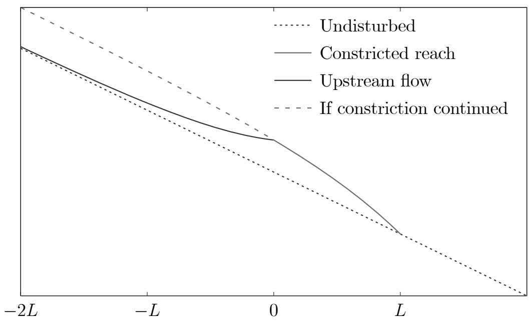
For x downstream, $x \geqslant L$, $\phi(x) = 0$, and $h = h_0$, which is correct in this sub-critical flow, there are no downstream effects.

For x in the section where the changes occur, $0 \leqslant x \leqslant L$, the solution is

$$h = h_0 + S_0\delta \int_x^L e^{\gamma(x-x')}\, dx' = h_0 + \frac{S_0\delta}{\gamma}\left(1 - e^{\gamma(x-L)}\right).$$

For x upstream, $x \leqslant 0$, where there is no extra resistance,

$$h = h_0 + S_0\delta e^{\gamma x} \int_0^L e^{-\gamma x'}\, dx' = h_0 + \frac{S_0\delta}{\gamma} e^{\gamma x}\left(1 - e^{-\gamma L}\right).$$



These solutions are all shown in the figure with an arbitrary vertical scale such that the slope is exaggerated. The calculations were performed for $S_0 = 0.0005$, $h_0 = 1\,\mathrm{m}$, and with a constricted length of $L = 1000\,\mathrm{m}$, with a 10% increase in resistance there, such that $\delta = 0.1$. Using these figures, and with $\gamma = 3S_0/h_0$, the computed backwater at the beginning of the constriction calculated according to the formula was $2.6\,\mathrm{cm}$.

In the reach of increased resistance the surface is raised, as one expects and shows an exponential approach to the changed depth $S_0\delta/\gamma$ if $L \to \infty$.

The abrupt changes of gradient violate our physical assumptions of the long wave equations, but they give us a clear picture of what happens, possibly obvious in retrospect, but hopefully of assistance.

We have obtained an approximate solution to the problem, with little input data necessary.

9. Hydrometry - measurement and analysis

9.1 Definitions

In English, the traditional word used to describe the measurement of water levels and flow volumes is “Hydrography”. That is ambiguous, for that word is also used for the measurement of water depths for navigation purposes, has been so used since the great navigators of the eighteenth century. Organisations with names like “National Hydrographic Service” are usually only concerned with the mapping of an area of sea and surrounding coastal detail.

Here we follow Boiten (2003) and Morgenschweis (2010) who provide a refreshingly modern approach to the topic, calling it “Hydrometry”, the “measurement of water”. In these notes, a practitioner will be called a hydrometrician, but the term hydrograph will be retained for a record, either digital or graphical, of the variation of water level or flow rate with time.

Two modern documents from the World Meteorological Organization provide more background. Experimental techniques are described in WMO Manual 1 (2010), and methods of analysis in WMO Manual 2 (2010). It is remarkable, however, that a field so important has received little benefit from hydraulics research.

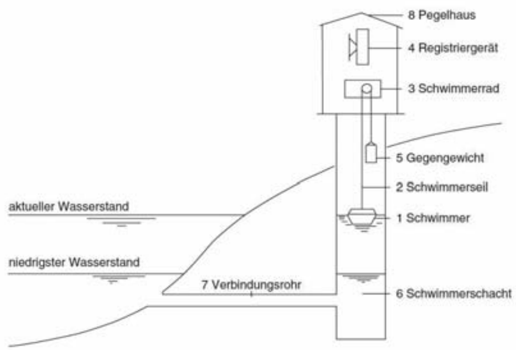
9.2 The Problem

Almost universally the routine measurement of the state of a river is that of the stage, the surface elevation at a gauging station. While that is an important quantity in determining the danger of flooding, another important quantity is the actual volume flow rate past the gauging station. Accurate knowledge of this instantaneous discharge - and its time integral, the total volume of flow - is crucial to many hydrologic investigations and to practical operations of a river and its chief environmental and commercial resource, its water. Examples include decisions on the allocation of water resources, the design of reservoirs and their associated spillways, the calibration of models, and the interaction with other computational components of a network.

Stage is usually simply measured. Measuring the flow rate, the discharge, is rather more difficult. Almost universally, occasionally (once per month, or more likely, once per year) it is obtained by measuring the velocity field in detail and integrating it with respect to area. At the same time, the water level is measured. This gives a pair of values (η_i, Q_i) which obtained on that day. Over a long period, a finite number of such data pairs are obtained using this laborious method. A curve that approximates those points is calculated, to give a function $Q_\mathrm{r}(\eta)$, a *Rating Curve*.

Separately, the actual stage can be measured easily and monitored almost continuously at any time, and automatically transmitted and recorded at intervals of one hour or one day, to give a *Stage Hydrograph*, a discrete representation of $\eta_n = \eta(t_n)$, $n = 0, 1, \ldots$. To get the corresponding *Discharge Hydrograph*, each value of η_n is considered and from the rating curve the corresponding $Q_n = Q_\mathrm{r}(\eta_n)$, $n = 0, 1, \ldots$ are calculated. Values of η_n and Q_n are published and made available.

9.3 Routine measurement of water levels



Stilling well & level recorder (Morgenschweis 2010)

Most water level gauging stations are equipped with a sensor or gauge plus a recorder. In many cases the water level is measured in a stilling well, thus eliminating strong oscillations.

Staff gauge:

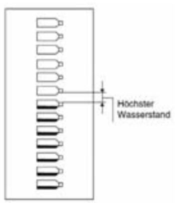
This is the simplest type, with a graduated gauge plate fixed to a stable structure such as a pile, bridge pier, or a wall. Where the range of water levels exceeds the capacity of a single gauge, additional ones may be placed on the line of the cross section normal to the plane of flow.

Float gauge:

A float inside a stilling well, connected to the river by an inlet pipe, is moved up and down by the water level. Fluctuations caused by short waves are almost eliminated. The movement of the float is transmitted by a wire passing over a float wheel, which records the motion, leading down to a counterweight.

Pressure transducers:

Water level is measured as hydrostatic pressure and transformed into an electrical signal *via* a semi-conductor sensor. These are best suited for measuring water levels in open water (the effect of short waves dies out almost completely within half a wavelength down into the water). They should compensate for changes in the atmospheric pressure, and if air-vented cables cannot be provided air pressure must be measured separately.



Peak level indicators:

There are some indicators of the maximum level reached by a flood, such as arrays of bottles which tip and fill when the water reaches them, or a staff coated with soluble paint.

Bubble gauge:

This is based on measurement of the pressure which is needed to produce bubbles through an underwater outlet. These are used at sites where it would be difficult to install a float-operated recorder or pressure transducer. From a pressurised gas cylinder or small compressor gas is led along a tube to some point under the water (which will remain so for all water levels) and small bubbles constantly flow out through the orifice. The pressure in the measuring tube corresponds to that in the water above the orifice. Wind waves should not affect this.

Ultrasonic sensor:

These are used for continuous non-contact level measurements in open channels. The sensor points

vertically down towards the water and emits ultrasonic pulses at a certain frequency. The inaudible sound waves are reflected by the water surface and received by the sensor. The round trip time is measured electronically and appears as an output signal proportional to the level. A temperature probe compensates for variations in the speed of sound in air. They are accurate but susceptible to wind waves.

9.4 Occasional measurement of discharge

Traditional manner of taking current meter readings. In deeper water a boat is used.

Most methods of measuring the rate of volume flow past a point are single measurement methods which are not designed for routine operation. Below, some will be described that are methods of continuous measurements.

Velocity area method ("current meter method")

The area of cross-section is determined from soundings, and flow velocities are measured using propeller current meters, electromagnetic sensors, or floats. The mean flow velocity is deduced from points distributed systematically over the river cross-section. In fact, what this usually means is that two or more velocity measurements are made on each of a number of vertical lines, and any one of several empirical expressions used to calculate the mean velocity on each vertical, the lot then being integrated across the channel.

Calculating the discharge requires integrating the velocity data over the whole channel - what is

required is the area integral of the velocity, that is $Q = \int u\, dA$. If we express this as a double integral we can write

$$Q = \int_B \int_{Z(y)}^{Z(y)+h(y)} u\, dz\, dy, \tag{9.1}$$

so that we must first integrate the velocity from the bed $z = Z(y)$ to the surface $z = Z(y) + h(y)$, where h is the local depth. Then we have to integrate these contributions across the channel, for values of the transverse co-ordinate y over the breadth B.

Calculation of mean velocity in the vertical

The first step is to compute the integral of velocity with depth, which hydrometricians think of as calculating the mean velocity over the depth. Consider the law for turbulent flow over a rough bed:

$$u = \frac{u_*}{\kappa} \ln \frac{z - Z}{z_0}, \tag{9.2}$$

where u_* is the shear velocity, $\kappa = 0.4$, $\ln()$ is the natural logarithm to the base e, z is the elevation above the bed, and z_0 is the elevation at which the velocity is zero. (It is a mathematical artifact that below this point the velocity is actually negative and indeed infinite when $z = 0$ – this does not usually matter in practice). If we integrate equation (9.2) over the depth h we obtain the expression

for the mean velocity:

$$\bar{u} = \frac{1}{h} \int_{Z(y)}^{Z(y)+h(y)} u\, dz = \frac{u_*}{\kappa} \left(\ln \frac{h}{z_0} - 1 \right). \tag{9.3}$$

Now it is assumed that two velocity readings are made, obtaining u_1 at z_1 and u_2 at z_2. This gives enough information to obtain the two quantities u_*/κ and z_0. Substituting the values for point 1 into equation (9.2) gives us one equation and the values for point 2 gives us another equation. Both can be solved to give the simple formula for the mean velocity in terms of the readings at the two points:

$$\bar{u} = \frac{u_1 (\ln(z_2/h)+1) - u_2 (\ln(z_1/h)+1)}{\ln (z_2/z_1)}. \tag{9.4}$$

As it is probably more convenient to measure and record depths rather than elevations above the bottom, let $h_1 = h - z_1$ and $h_2 = h - z_2$ be the depths of the two points, when equation (9.4) becomes

$$\bar{u} = \frac{u_1 (\ln(1 - h_2/h)+1) - u_2 (\ln(1 - h_1/h)+1)}{\ln ((h - h_2) / (h - h_1))}. \tag{9.5}$$

This expression gives the freedom to take the velocity readings at any two points. This would simplify streamgauging operations, for it means that the hydrometrician, after measuring the depth h, does not have to calculate the values of $0.2h$ and $0.8h$ and then set the meter at those points, as is done in current practice. Instead, the meter can be set at any two points, within reason, the depth and the velocity simply recorded for each, and equation (9.5) applied. This could be done either *in* *situ* or later when the results are being processed. This has the potential to speed up hydrographic measurements.

If the hydrometrician were to use the traditional two points, then setting $h_1 = 0.2h$ and $h_2 = 0.8h$ in equation (9.5) gives the result

$$\bar{u} = 0.4396\, u_{0.2h} + 0.5604\, u_{0.8h} \approx 0.44 u_{0.2h} + 0.56 u_{0.8h}\,, \tag{9.6}$$

whereas the conventional hydrographic expression is

$$\overline{u} = \tfrac{1}{2}\left(u_{0.2h} + u_{0.8h}\right), \tag{9.7}$$

that is, the mean of the readings at 0.2 of the depth and 0.8 of the depth. The nominally more accurate expression is just as simple as the traditional expression in a computer age, yet is based on an exact analytical integration of the equation for a turbulent boundary layer.

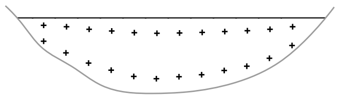


Figure 9.1: Cross-section of stream, showing velocity measurement points

The formula (9.6) has been tested by taking a set of gauging results. A canal had a maximum depth of 2.6m and was 28m wide, and a number of verticals were used. The conventional formula (9.7), the mean of the two velocities, was accurate to within 2% of equation (9.6) over the whole range of the readings, with a mean difference of 1%. That error was always an overestimate. The more accurate formula (9.5) is hardly more complicated than the traditional one, and it should in general be preferred. Although the gain in accuracy was slight in this example, in principle it is desirable to use an expression which makes no numerical approximations to that which it is purporting to

evaluate. This does not necessarily mean that either (9.7) or (9.6) gives an accurate integration of the velocities which were encountered in the field. In fact, one complication is where, as often happens in practice, the velocity distribution near the surface actually bends back such that the maximum velocity is below the surface.

Integration of the mean velocities across the channel

The problem now is to integrate the readings for mean velocity at each station across the width of the channel. Here traditional standards commit an error – often the *Mean-Section* method is used. In this the mean velocity between two verticals is calculated and then this multiplied by the area between them, so that, given two verticals i and $i+1$ separated by b_i the expression for the contribution to discharge is assumed to be

$$\delta Q_i = \frac{1}{4} b_i\, \left(h_i + h_{i+1}\right)\left(\overline{u}_i + \overline{u}_{i+1}\right).$$

This is not correct. From equation (9.1), the task is actually to integrate across the channel the quantity which is the mean velocity times the depth. For that the simplest expression is the Trapezoidal rule:

$$\delta Q_i = \frac{1}{2} b_i\, \left(\overline{u}_{i+1}\, h_{i+1} + \overline{u}_i\, h_i\right)$$

To examine where the Mean-Section Method is worst, we consider the case at one side of the channel, where the area is a triangle. We let the water's edge be $i = 0$ and the first internal point be

$i = 1$, then the Mean-Section Method gives

$$\delta Q_0 = \frac{1}{4} b_0 \overline{u}_1 h_1,$$

while the Trapezoidal rule gives

$$\delta Q_0 = \frac{1}{2} b_0 \overline{u}_1 h_1,$$

which is correct, and we see that the Mean-Section Method computes only half of the actual contribution. The same happens at the other side. Contributions at these edges are not large, and in the middle of the channel the formula is not so much in error, but in principle the Mean-Section Method is wrong and should not be used. Rather, the Trapezoidal rule should be used, which is just as easily implemented. In a gauging in which the lecturer participated, a flow of $19.60\ \mathrm{m^3s^{-1}}$ was calculated using the Mean-Section Method. Using the Trapezoidal rule, the flow calculated was $19.92\ \mathrm{m^3s^{-1}}$, a difference of 1.6%. Although the difference was not great, practitioners should be discouraged from using a formula which is wrong.

An alternative global "spectral"approach with least-squares fitting

It is strange that only very local methods are used in determining the vertical velocity distribution. Here we consider a significant generalisation, where we consider velocity distributions given by a more general law, assuming an additional linear and an additional quadratic term in the velocity profile:

$$u(y, z) = a_0(y) \ln \frac{z - Z(y)}{z_0} + a_1(y)\ z + a_2(y)\, z^2, \tag{9.8}$$

but where the coefficients $a_0(y)$, $a_1(y)$, and $a_2(y)$ are actually polynomials in the transverse co-ordinate. The whole expression is a global function, that approximates the velocity over the whole section. If the polynomials in y each have J terms, then the total number of unknown coefficients is $3J$. Consider a number of flow measurements U_n for $n = 1$ to N, where we presume that the corresponding bed elevation is also measured, however that is not essential to the method. We compute the total sum of the errors squared, using equation (9.8):

$$\varepsilon = \sum_{n=1}^{N} \left(u\left(y_n, z_n\right) - U_n\right)^2,$$

and we use package software to find the coefficients such that the total error ε is minimised. Or, as one says, the function is "fitted to the data". This method does not require points to be in vertical lines, although it is often convenient to measure points like that, as well as the corresponding bed level. An example of the results is given in figure 9.2, where more than two points were used on each vertical line. It can be seen that results are good - and we see an import feature that the conventional method ignores – almost everywhere there is a velocity maximum in the vertical.

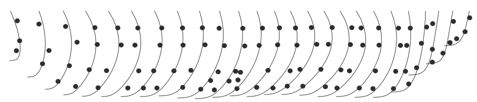


Figure 9.2: Cross-section of canal with velocity profiles and data points plotted transversely, showing fit by global function

Dilution methods

In channels where cross-sectional areas are difficult to determine (*e.g.* steep mountain streams) or where flow velocities are too high to be measured by current meters *dilution* or *tracer* methods can be used, where continuity of the tracer material is used with steady flow. The rate of input of tracer is measured, and downstream, after total mixing, the concentration is measured. The discharge in the stream immediately follows.

Ultrasonic flow measurement

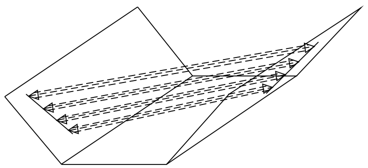


Figure 9.3: Array of four ultrasonic beams in a channel

In this case, sound generators are placed along the side of a channel and beamed so as to cross it diagonally. They are reflected on the other side, and the total time of travel of the sound waves are measured. From that it is possible to calculate the mean water velocity along the channel – the sound "samples" the water velocity at all points. Then, to get the total discharge it is necessary to integrate the mean velocity of the paths in the vertical. This, unfortunately, is where the story ends unhappily. The performance of the trade and scientific literature has been poor. Several trade brochures advocate the routine use of a single beam, or maybe two, suggesting that that is adequate (see, for example, Boiten 2003, p141). In fact, with high-quality data for the mean velocity at two or three levels, there is no reason not to use accurate integration formulae. However, practice in this area has been quite poor, as

trade brochures that the author has seen use the inaccurate Mean-Section Method for integrating vertically over only three or four data points, when its errors would be rather larger than when it is used for many verticals across a channel, as described previously. The lecturer found that no-one wanted to know of his discovery.

Acoustic-Doppler Current Profiling (ADCP) methods

In these, a beam of sound of a known frequency is transmitted into the fluid, often from a boat. When the sound strikes moving particles or regions of density difference moving at a certain speed, the sound is reflected back and received by a sensor mounted beside the transmitter. According to the Doppler effect, the difference in frequency between the transmitted and received waves is a direct measurement of velocity. In practice there are many particles in the fluid and the greater the area of flow moving at a particular velocity, the greater the number of reflections with that frequency shift. Potentially this method is very accurate, as it purports to be able to obtain the velocity over quite small regions and integrate them up. However, this method does not measure in the top 15% of the depth or near the boundaries, and the assumption that it is possible to extract detailed velocity profile data from a signal seems to be optimistic. The lecturer remains unconvinced that this method is as accurate as is claimed.

Electromagnetic methods

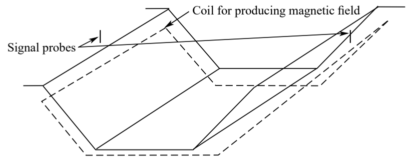


Figure 9.4: Electromagnetic installation, showing coil and signal probes

The motion of water flowing in an open channel cuts a vertical magnetic field which is generated using a large coil buried beneath the river bed, through which an electric current is driven. An electromotive force is induced in the water and measured by signal probes at each side of the channel. This very small voltage is directly proportional to the average velocity of flow in the cross-section. This is particularly suited to measurement of effluent, water in treatment works, and in power stations, where the channel is rectangular and made of concrete; as well as in situations where there is much weed growth, or high sediment concentrations, unstable bed conditions, backwater effects, or reverse flow. This has the advantage that it is an integrating method, however in the end recourse has to be made to empirical relationships between the measured electrical quantities and the flow.

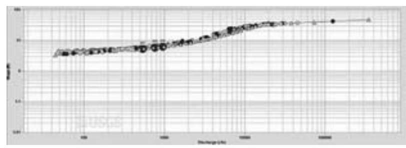
9.5 Rating curves – the analysis and use of stage and discharge measurements

The state of the art – the power function

The generation of rating curves from data is a problem that is of crucial importance, but has had little research attention and is done very badly all around the world. Almost universally it is believed that they *must* follow a power function

$$Q = C\,(h - h_0)^{\mu}\,, \tag{9.9}$$

where h is water surface elevation (stage), C is a constant, h_0 is a constant elevation reference level, for zero flow, and μ is a constant with a typical value in the range 1.5 to 2.5.



US Geological Survey - fitting three straight lines to data segments

If we take logarithms of both sides of equation (9.9),

$$\log Q = \log C + \mu \log (h - h_0)\,, \tag{9.10}$$

then on a figure plotting $\log Q$ against $\log (h - h_0)$, a straight line is obtained. Often in practice, the data is divided and different straight-line approximations are used, as in the figure.

A problem is that h_0, the nominal zero flow point, is not initially known and has to be found, which is actually a difficult nonlinear problem. A larger problem is that the equation is only a rough approximation, but because of its ability occasionally to describe roughly almost all of a rating curve, it has acquired an almost-sacred status, and far too much attention has been devoted to it rather than addressing the problems of how to approximate rating data generally and accurately. Much modelling and computer software follow its dictates. It really has been believed to be a "law".

The reason that the power law has been believed to be so powerful (sorry) is that hydrologists believe the hydraulic formula – and it does describe the discharge of a sharp-crested infinitely-wide weir in water of infinite depth and the steady uniform flow in an infinitely-wide channel. We know enough hydraulics to know that not all rivers satisfy those conditions ... and we treat the problem as one, not of hydraulics, but of *data approximation*, because the hydraulics are complicated.

The hydraulics of a gauging station

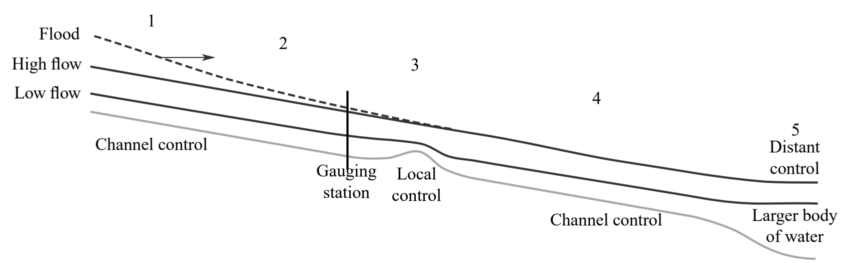


Figure 9.5: Section of river showing different controls at different water levels with implications for the stage discharge relationship at the gauging station shown

Local control: Just downstream of the gauging station is often some sort of fixed control which may be some local topography such as a rock ledge which means that for relatively small flows there is a definite relationship between the head over the control and the discharge, similar to a weir. This will control the flow for small flows.

Channel control: For larger flows the effect of the fixed control is to "drown out", to become unimportant, and where the control is due to resistance in the channel.

Overbank control: For larger flows when the river breaks out of the main channel and spreads onto the surrounding floodplain, the control is also due to resistance, but where the geometry of channel and nature of the resistance is different.

Distant control: There may be something such as the larger river downstream shown as a distant control in the figure. In our work on Steady Flow, we saw that backwater influence can extend for a long way.

In practice, the natures of the controls are unknown.

Data approximation

The global representation of Q by a polynomial has been in the background for some time:

$$Q = a_0 + a_1 h + a_2 h^2 + \ldots + a_M h^M = \sum_{m=0}^{M} a_m h^m \,, \tag{9.11}$$

where a_0, a_1, $\ldots$, a_M are coefficients. Standard linear least-squares methods can be used to determine the coefficients, but it has never really succeeded.

Fenton & Keller (2001, §6.3.2) suggested writing the polynomial for Q raised to the power ν, specified *a priori*:

$$Q^\nu = a_0 + a_1 h + a_2 h^2 + \ldots + a_M h^M = \sum_{m=0}^{M} a_m h^m \,, \tag{9.12}$$

which is a simple generalisation of the power function to $Q = \left(a_0 + a_1 h + a_2 h^2 + \ldots\right)^{1/\nu}$. A value of $\nu = \frac{1}{2}$ was recommended as that was the mean value in the hydraulic discharge formulae for a sequence of weir and channel cross-sections that modelled local and channel control.

The use of such a fractional value has two effects:

1. For small flows, h small, the data usually is such that

$$Q^\nu = a_0 + a_1 h, \tag{9.13}$$

   that is

$$Q = (a_0 + a_1 h)^{1/\nu} = C\,(h - h_0)^\mu \,,$$

   so it looks like the simple power law! In fact, $\nu = \frac{1}{2}$ is a good approximation. In that low-flow limit the polynomial just has to simulate nearly-linear variation, which it can easily do.
2. For large flows, the use of $Q^{1/2}$ means that the magnitude of the dependent variable to be approximated is much smaller, so that, instead of a range of say, $Q = 1$ to $10^4\ \mathrm{m^3 s^{-1}}$, a numerical range 1 to 10^2 has to be approximated.

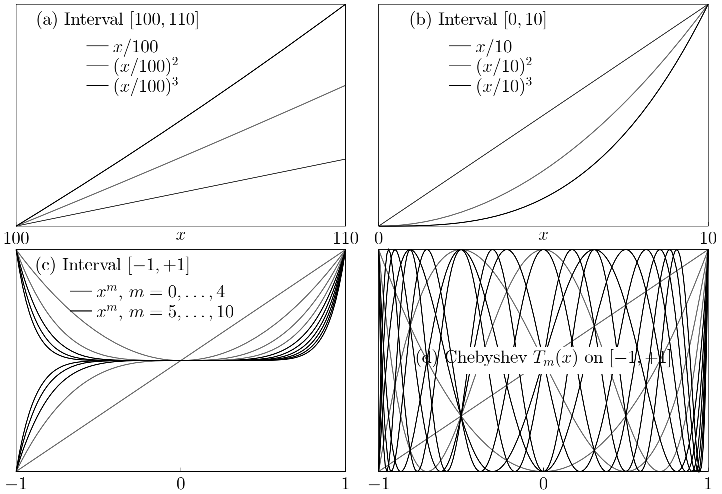
The lecturer (Fenton 2015*b*) has shown it is better to generalise equation (9.12) by considering the approximating function to be made up, not of monomials h^m, but more general Chebyshev functions

$$Q^\nu = \sum_{m=0}^{M} a_m\, T_m\,(y) \ . \tag{9.14}$$

With these modifications, global approximation is more stable and accurate.

Problems with global interpolation and approximation

The simplest set of basis functions are the *monomials* $p_m(x) = x^m$. They are not very good, as they all look rather like each other for large x and for $m = 2$ or greater. Individual basis functions $p_m(x)$ should look different from each other so that irregular variation can be described efficiently.



Least-squares approximation

The coefficients a_m can be obtained by least-squares software, minimising the sum of the weighted squares of the errors of the approximation over N data points,

$$\varepsilon_2 = \sum_{n=1}^{N} w_n \left(\sum_{m=0}^{M} a_m \, T_m(y_n) - Q_n^{\nu} \right)^2 ,$$

where the y_n are obtained from the h_n by scaling the range of all stage measurements $[h_{\min}, h_{\max}]$. The w_n are the weights for each rating point, giving the freedom to weight some points more if one wanted the rating curve to approximate them more closely, or they could be set to be a decaying function of the age of the data point, so that the effects of changes with time could be examined. Or, a less-trusted data point could be given a smaller weight. Often, however, all the w_n will be 1.

An example

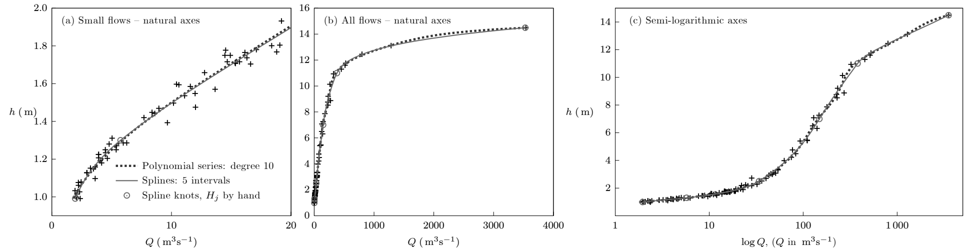


Figure 9.6: Noxubee River near Geiger, AL, USA, USGS Station 02448500, 1970s

- The example has quite a striking ideal form showing local, channel and overbank control.
- Variation at the low flow end can be rapid, and can have a vertical gradient and high curvature.
- The discharge may extend over 3 or 4 orders of magnitude (factor of 1000 or 10000)
- There can be rapid variation between these different regimes
- There are two curves that have been fitted by least-squares methods, one using Chebyshev polynomials, the other using local spline approximation. Both have worked well.

Possible problems

There are several problems associated with the use of a Rating Curve:

- Discharge is rarely measured during a flood, and the quality of data at the high flow end of the curve might be quite poor.
- There are a number of factors which might cause the rating curve not to give the actual discharge, some of which will vary with time. Factors affecting the rating curve include:
  – The channel changing as a result of modification due to dredging, bridge construction, or vegetation growth.
  – Sediment transport - where the bed is in motion, which can have an effect over a single flood event, because the effective bed roughness can change during the event. As a flood increases, any bed forms present will tend to become larger and increase the effective roughness, so that friction is greater after the flood peak than before, so that the corresponding discharge for a given stage height will be less after the peak.
  – Backwater effects - changes in the conditions downstream such as the construction of a dam or flooding in the next waterway.
  – Unsteadiness - in general the discharge will change rapidly during a flood, and the slope of the water surface will be different from that for a constant stage, depending on whether the discharge is increasing or decreasing, also contributing to a flood event appearing as a loop on a stage-discharge diagram.
  – Variable channel storage - where the stream overflows onto flood plains during high discharges, giving rise to different slopes and to unsteadiness effects.

Modelling rating curve changes with time

The importance of each data point can be weighted according to their age, so that the oldest points have the smallest contributions to the least squares error, and the most recent gaugings can be rationally incorporated to give the most recent rating curve. In fact, the rating curve can be constructed for any day, now or in the past.

We use a smooth function, decaying into the past, to keep all the points to some extent in determining the shape of the curve, the exponential weight factor $w(\tau) = \exp(-\alpha\tau)$, where α is a decay constant. Writing $\tau_{1/2}$ for the "half-life", the age at which the weight decays by a factor of $\frac{1}{2}$, then the expression becomes

$$w(t_0 - t_n) = \left(\frac{1}{2}\right)^{(t_0-t_n)/\tau_{1/2}},$$

where t_0 is the date for which the rating curve is required, and t_n is the date when point n was established. If $t_n > t_0$ a value of zero is used. This was applied to 31 years of data from USGS Station 02448500 on the Noxubee River near Geiger, AL, USA, shown in Fig. 9.7. The approximating spline method with 6 hand-allocated intervals was used, with a $Q^{1/2}$ fit. It can be seen how the rating curve, and presumably the bed, has moved down over the 31 years.

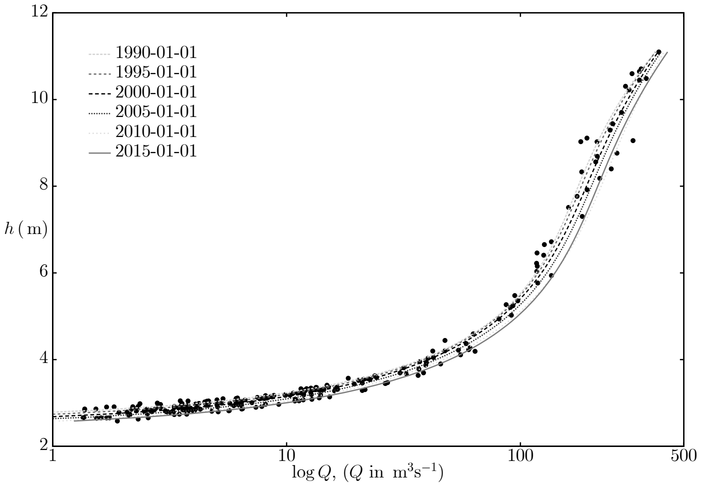


Figure 9.7: Calculation of rating curves on specific days using weights that are a function of measurement age — Noxubee River near Geiger, AL, USA, USGS Station 02448500, from 1984-10-02 to 2015-05-11

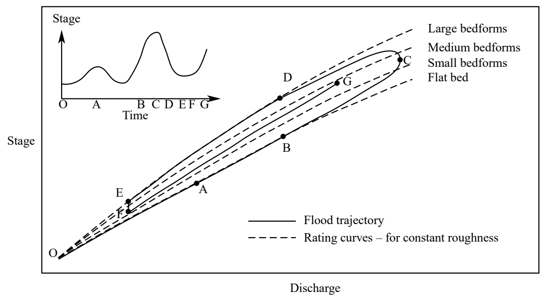
The effects of bed roughness and bed changes on rating curves in alluvial streams

If there is a rapidly-changing flow event such as a flood, roughness and hence resistance might also change relatively quickly, and the relationship between stage and discharge changes with time such that if we were able to measure and plot it throughout the unsteady event we would obtain a looped curve with two discharges for each stage, and *vice versa*, before and after the flow maximum. This is usually described as a *looped rating curve*. The lecturer has usually been sceptical of that term, viewing it more as a *looped flow trajectory on rating axes*.

Let us consider the mechanism by which changes in resistance cause the flood trajectories to be looped, by considering a hypothetical and idealised situation. We do not know how much bed-forms and how much individual grains are responsible for most resistance.

The figure on the next page is plotted with rating curve axes, stage versus discharge. The rating curves which would apply if the resistance were a particular value are shown, for a flat bed with co-planar grains, and for various increasing resistances.

In the top left corner is a stage-time graph with two flood events, and another not yet completed. The points labelled O, A, ..., G are also shown on the *flow trajectory*, showing the actual relationship between stage and discharge at each time.



O: flow is low, over a flat co-planar bed after a period of steady flow.

A: the flow increases. The flow is not enough to change the nature of the bed, and the flood trajectory follows the flat-bed rating curve up to here.

B: the bed is no longer stable, grains move and bed forms develop. Accordingly, the resistance is greater and the stage increases.

C: flood peak has arrived, resistance continues to increase, a little later the stage is a maximum.

D: resistance and bedforms have continued to grow until here, although flow is decreasing.

E: the flow has decreased much more quickly than the bed can adjust, and the point is close to the instantaneous rating curve corresponding to the greatest resistance.

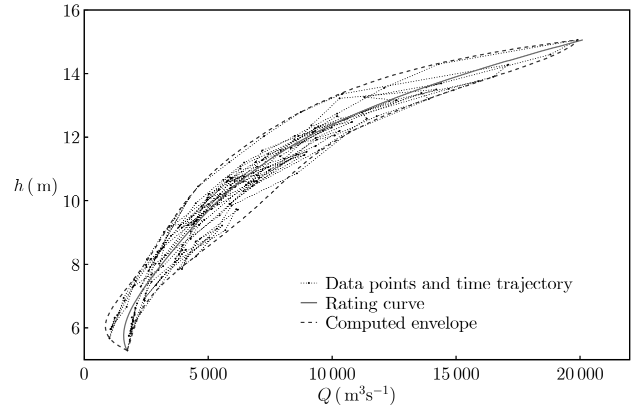
F: over the intervening time, flow has been small and almost constant, however the time has been enough to reduce the bed-forms and pack the bed grains to some extent. Now another flood starts to arrive, and this time, instead of following the flat-bed curve, it already starts from a finite resistance.

G: after this, the history of the stage will still depend on the history of the flow and the characteristics of the rate of change of the bed.

The rating trajectory and rating envelope — generalisations of the single curve

- The ephemeral nature of the resistance means that in highly mobile bed streams it is not possible to calculate accurately the flow at any later time. It is not known what flows and bed changes will occur in the future up until the moment the curve is required to give a flow from a routine stage measurement.
- The view of the rating curve here is that it is an approximate curve passing through a more-or-less scattered cloud of points, where at least some of that scatter is due to fluctuations in the preceding flows and instantaneous state of the bed when each point was determined.
- If a long period of time is considered, with many flow events of different magnitudes, the flow trajectory will consist of a number of different paths and loops, the whole adding up to a complicated web occupying a limited region.
- Usually, however, one does not measure rating points very often, and instead of a continuous flow trajectory following an identifiable path, one just sees a discrete number of apparently-random points, occupying a more-or-less limited region.
- The more stable the bed, the less will be the scatter.
- Generally the points will fall in a band between a lower boundary, corresponding to smaller resistance, with an armoured bed, and an upper boundary corresponding to greater resistance with individual grains protruding and possibly bed-forms prominent when, for a given flow, the water will be deeper and stage higher.

- This leads to an extension of the idea of a single rating curve: that in a stream of variable bed conditions, one can never know the situation when rating data is actually to be used to predict a flow, and so it might be helpful also to compute a *rating envelope*, to provide expressions for curves approximating both upper and lower bounds.
- Suggested procedure:
  - Calculate the approximation to all the points, the rating curve
  - Then delete those points which lie *below* it.
  - Approximate the remaining points, and repeat as many times as necessary, to give the upper envelope.
  - Repeat, successively deleting all points *above* each curve.
  - As approximately half the data points are lost with each pass, the number of passes is limited. In practice what one would be doing is approximating the 1/8 or 1/16, say, of all data points, those which lie furthest from the approximation to all the points.



The figure shows an example for three years (1995–1997) of gaugings from Station 41 on the Red River, Viet Nam. The flow trajectory has several large loops, barely visible in the figure. Four passes of the halving procedure for each of the upper and lower envelopes were applied, starting with 217 data points, at the end there were about $217/2^4 \approx 15$ for each envelope. It can be seen that the method worked well.

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