Hydraulics

Solution Sheet 6 – The momentum theorem

1. *A jet of water issues from a nozzle at a speed of* $6\,\mathrm{m\,s^{-1}}$ *and strikes a stationary flat plate oriented perpendicular to the jet. The exit area of the nozzle is* $645\,\mathrm{mm}^2$ *and the Bousinesq momentum coefficient is* $\beta = 1.05$. *Draw a diagram of the flow plus a control volume and calculate the total force on the plate from the fluid in contact with it for two cases*

   a. *if the fluid travels parallel to the plate after impact, and*

   b. *if each particle of fluid rebounds back in the direction from which it came without loss?*

   (Ans: $24.4\,\mathrm{N}$*, Twice that).*

   We can use the formula given in the lecture notes. If you are uncertain, then do it from first principles.

   a. The formula derived is

$$\begin{aligned} P &= +\rho\beta_1 U_1^2 A_1 \\ &= 1000 \times 1.05 \times 6^2 \times \frac{645}{10^6} \\ &= 24.4\,\mathrm{N} \end{aligned}$$

   b. As the jet is in air, the pressure is atmospheric throughout and we are given no information about elevation differences which will be small anyway, so that we can use the energy theorem and conclude that the speed of the water travelling in the other direction is the same, and by mass conservation the cross-sectional area will be the same. Hence the momentum is completely reversed rather than just brought to zero in the horizontal, and the force will be twice, $48.8\,\mathrm{N}$.

2. *A flat plate is struck normally by a jet of water* $50\,\mathrm{mm}$ *in diameter with a velocity of* $18\,\mathrm{m\,s^{-1}}$ and $\beta = 1.05$. *Calculate*

   a. *the force on the plate when it is stationary, and*

   b. *the force when it moves in the same direction as the jet with a velocity of* $6\,\mathrm{m\,s^{-1}}$. *(To do this you will have to superimpose a uniform horizontal velocity to your system such that all motion is steady)*

   (Ans. $668\,\mathrm{N}$, $283\,\mathrm{N}$.*)*

   This is the same sort of problem as the previous one

   Part (a) is exactly the same sort:

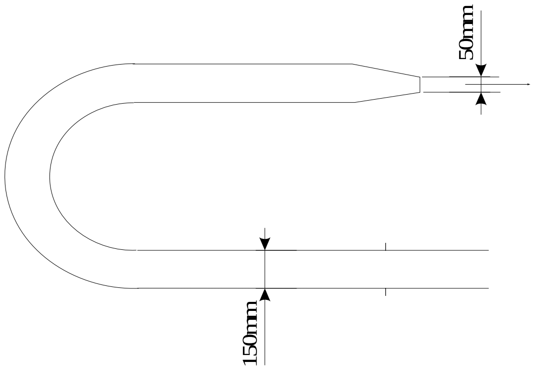
$$\begin{aligned} P &= +\rho\beta_1 U_1^2 A_1 \\ &= +1000 \times 1.05 \times 18^2 \times \frac{\pi}{4}0.05^2 \\ &= 668\,\mathrm{N} \end{aligned}$$

   Part (b) is actually an unsteady problem when viewed by a stationary observer in our inertial frame. To make the problem steady so that we can apply the steady momentum theorem we superimpose a horizontal velocity of $-6\,\mathrm{m\,s^{-1}}$ to the system such that the plate is now stationary and the incoming fluid has a velocity of $12\,\mathrm{m\,s^{-1}}$, when we can apply the formulae developed in the previous two problems:

$$\begin{aligned} P &= +\rho\beta_1 U_1^2 A_1 \\ &= +1000 \times 1.05 \times 12^2 \times \frac{\pi}{4}0.05^2 \\ &= 297\,\mathrm{N} \end{aligned}$$

3. *The pipe bend and nozzle in the figure is bolted onto a pipe at 1, where it has a diameter of* $150\,\mathrm{mm}$.

It turns $180°$ *in a horizontal plane and narrows down to* $50\,\mathrm{mm}$ *as it discharges the water into air. The discharge is* 56.5 *Ls*$^{-1}$.



a. *Calculate mean velocity at 1 and at 2.*

$$\begin{aligned}
Q &= 56.5\,\mathrm{L\,s^{-1}} = 0.0565\,\mathrm{m^3\,s^{-1}} \\
A_1 &= \frac{\pi}{4} \times 0.15^2 = 0.0177\,\mathrm{m^2} \\
U_1 &= \frac{0.0565}{0.0177} = 3.19\,\mathrm{m\,s^{-1}} \\
A_2 &= \frac{\pi}{4} \times 0.05^2 = 0.00196\,\mathrm{m^2} \\
U_2 &= \frac{0.0565}{0.00196} = 28.8\,\mathrm{m\,s^{-1}}
\end{aligned}$$

It is very helpful to reduce these all to strict SI units: metres, seconds etc.

b. *Calculate the pressure at 1. Assume* $\alpha = 1.3$ *in the energy equation.*

We can use the energy equation in the form

$$\begin{aligned}
\left(\frac{p}{\rho g} + z + \frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_1 &= \left(\frac{p}{\rho g} + z + \frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_2 \\
\frac{p_1}{\rho g} + 0 + \frac{1.3}{2 \times 9.8} 3.19^2 &= 0 + 0 + \frac{1.3}{2 \times 9.8} 28.8^2 \\
p_1 &= 1000 \times 9.8 \times \frac{1.3}{2 \times 9.8} \times \left(28.8^2 - 3.19^2\right) \\
&= 532\,\mathrm{kPa}
\end{aligned}$$

c. *Calculate the net horizontal force exerted on the pipe bend and nozzle. Assume* $\beta = 1.15$ *in the momentum equation.*

We have from the lecture notes (Last year's tutors – this is a much simpler formulation which I discovered):

$$\mathbf{P} = \sum_j \left(\rho \beta_j U_j^2 A_j + \bar{p}_j A_j\right)(-\hat{\mathbf{n}}_j) + \mathbf{F}_{\text{body}}$$

In this case we can make the usual pipe approximation that pressure is constant across it. Hence we have, just considering forces in the horizontal direction, such that $\hat{\mathbf{n}}_1 = +\mathbf{i}$, and also $\hat{\mathbf{n}}_2 =$

$+\mathbf{i}$ so that we have

$$
\begin{aligned}
P\mathbf{i} &= \left(\rho\beta_1\frac{Q_1^2}{A_1}+\bar{p}_1A_1+\rho\beta_2\frac{Q_2^2}{A_2}+\bar{p}_2A_2\right)(-\mathbf{i})\\
P &= -\left(\rho\beta_1\frac{Q_1^2}{A_1}+\bar{p}_1A_1+\rho\beta_2\frac{Q_2^2}{A_2}+\bar{p}_2A_2\right)\\
&= -\left(1000\times 1.15\times\frac{0.0565^2}{0.0177}+532000\times 0.0177+1000\times 1.15\times\frac{0.0565^2}{0.00196}+0\right)\\
&= -11500\,\mathrm{N}=-11.5\,\mathrm{kN}
\end{aligned}
$$

Thus, the force of the fluid on the body is $11.5\,\mathrm{kN}$ *to the left.* Note that all contributions conspire together in this problem – the incoming fluid moving to the left has to be stilled and then accelerated to the right, with both giving contributions to the force to the left, and the pressure forces on the first end is to the left (the other is zero).

d. *Repeat (b) and (c) using* $\beta=\alpha=1$.

$$
\begin{aligned}
p_1 &= 1000\times 9.8\times\frac{1.0}{2\times 9.8}\times\left(28.8^2-3.19^2\right)\\
&= 410\,\mathrm{kPa}\\
P &= -\left(1000\times 1.0\times\frac{0.0565^2}{0.0177}+410000\times 0.0177+1000\times 1.0\times\frac{0.0565^2}{0.00196}+0\right)\\
&= -9070\,\mathrm{N}=-9.07\,\mathrm{kN}
\end{aligned}
$$

Thus the force of fluid on the body is $9.1\,\mathrm{kN}$ to the left. It is interesting that the Coriolis and Boussinesq coefficients have quite a large contribution. I do not know why textbooks pretend that they are all 1.

4. *Water flows along a rectangular irrigation canal of width b and depth d, with discharge Q, Boussinesq momentum coefficient β, density ρ and gravitational acceleration g. You may assume that the pressure in the water is hydrostatic, as the streamlines are all parallel.*

   a. *Show that the magnitude of the momentum flux is*

$$\rho\left(\beta\frac{Q^2}{bd}+\tfrac{1}{2}gbd^2\right).$$

   b. *Check that your equation is dimensionally homogeneous.*

a. The expression for momentum flux is $\int_A(\rho\mathbf{u}\,\mathbf{u}\cdot\hat{\mathbf{n}}+p\,\hat{\mathbf{n}})\,dA$. The inertial flux component we simply approximate with the aid of the Boussinesq coefficient by $\rho\beta U^2A\mathbf{i}$, where $\mathbf{i}$ is a unit vector along the channel. However, $U=Q/A=Q/bd$ and $A=bd$, hence we have the contribution $\rho\beta Q^2/A\,\mathbf{i}=\rho\beta Q^2/(bd)\,\mathbf{i}$.

Now for the pressure component, we assume it is hydrostatic, such that if the z origin is on the bed, $p=\rho g(d-z)$. Integrating, with $\hat{\mathbf{n}}=\mathbf{i}$,

$$
\begin{aligned}
\textstyle\int_A p\,\hat{\mathbf{n}}\,dA &= \rho gb\int_0^d(d-z)\,dz\;\mathbf{i}\\
&= \rho gb(dz-z^2/2)\big|_0^d\;\mathbf{i}\\
&= \rho gbd^2/2\;\mathbf{i},
\end{aligned}
$$

and combining the two contributions

$$\rho\beta Q^2/(bd)\;\mathbf{i}+\rho gbd^2/2\;\mathbf{i},$$

and the magnitude is as we were required to prove.

b. Dimensional check – the ρ is a constant factor, we will not include it:

$$\beta\frac{Q^2}{bd}+\tfrac{1}{2}gbd^2$$
$$\left(\frac{L^3}{T}\right)^2\frac{1}{L^2}=\frac{L^4}{T^2},\quad LT^{-2}L^3=\frac{L^4}{T^2},$$

thus it is OK.

5. *The force on the nozzle of a fire hose - a fire hose is* $100\,\mathrm{mm}$ *in diameter and is required to deliver a stream of* $20\ \mathrm{L\,s^{-1}}$. *A nozzle is fixed to the hose which forces the jet of water to leave the hose with a diameter of* $50\,\mathrm{mm}$. *Use momentum principles to calculate the force on the nozzle. You may assume that the density of water is* $1000\,\mathrm{kg\,m^{-3}}$, *that the Boussinesq coefficient is* $\beta = 1.2$ *and the Coriolis coefficient* $\alpha = 1.3$.

Consider the momentum theorem:

$$\mathbf{P}=\sum_j\left(\rho\beta_j\frac{Q_j^2}{A_j}+\bar{p}_jA_j\right)(-\hat{\mathbf{n}}_j)+\mathbf{F}_{\text{body}}$$

In our case, if we consider the nozzle to be the control volume, which we assume to be pointed to the right in the direction of the unit vector **i**. The unit outward normal vector at the entry to the control volume (which we will call section 1) is then $\hat{\mathbf{n}}_1=-\mathbf{i}$, and at the exit it is $\hat{\mathbf{n}}_2=+\mathbf{i}$. Ignoring body forces (vertical, due to gravity):

$$\left(\rho\beta\frac{Q^2}{A_1}+p_1A_1\right)(-\mathbf{i})+\left(\rho\beta\frac{Q^2}{A_2}+p_2A_2\right)(+\mathbf{i})+P\mathbf{i}=\mathbf{0}.$$

However, $p_2=0$ (atmospheric), therefore

$$P=p_1A_1+\rho\beta Q^2\left(\frac{1}{A_1}-\frac{1}{A_2}\right). \tag{1}$$

However we don't yet know p_1, so that we will have to use the energy theorem between sections 1 and 2:

$$\left(\frac{p}{\rho g}+z+\frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_1=\left(\frac{p}{\rho g}+z+\frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_2,$$

and as $z_2=z_1$ and $p_2=0$,

$$\frac{p_1}{\rho}=\frac{\alpha}{2}\frac{Q^2}{A_2^2}-\frac{\alpha}{2}\frac{Q^2}{A_1^2}=\frac{\alpha Q^2}{2}\left(\frac{1}{A_2^2}-\frac{1}{A_1^2}\right). \tag{2}$$

Now we could substitute the expression for p_1 into equation (1) algebraically, but the result is more complicated than necessary. Instead firstly we will evaluate p_1. We have $\beta=1.2$, $A_1=\pi\times0.1^2/4=7.85\times10^{-3}\,\mathrm{m}^2$, $A_2=\pi\times0.05^2/4=1.96\times10^{-3}\,\mathrm{m}^2$. Evaluating equation (2):

$$\begin{aligned}p_1&=\rho\frac{\alpha}{2}Q^2\left(\frac{1}{A_2^2}-\frac{1}{A_1^2}\right)\\&=1000\times\frac{1.3}{2}\times\left(\frac{20}{1000}\right)^2\left(\frac{1}{(1.96\times10^{-3})^2}-\frac{1}{(7.85\times10^{-3})^2}\right)\\&\approx 63500\,\mathrm{Pa}\end{aligned}$$

Substituting into (1):

$$\begin{aligned}P&=+63500\times7.85\times10^{-3}+1000\times1.2\times\left(\frac{20}{1000}\right)^2\left(\frac{1}{7.85\times10^{-3}}-\frac{1}{1.96\times10^{-3}}\right)\\&=498.5-183.75\approx315\,\mathrm{N}.\end{aligned}$$

– As this is positive, the force on the nozzle is to the right, and the coupling between hose and nozzle has to be designed for tension.

- How large is $315\,\mathrm{N}$? Well, a human being of mass $80\,\mathrm{kg}$ has a gravitational force of $80 \times 9.8 \approx 780\,\mathrm{N}$, so it is approaching half the weight of a person. Presumably this means that the nozzle on the end of the hose would try to pull the hose out straight, and if there is nothing anchoring it, then the firemen must. It sounds like a challenge.
- Note that the contribution of the pressure force at 1 was positive, as expected, but that the contribution of the velocity difference to the force was negative. Is that right? Well, the momentum of the fluid leaving the nozzle is $\rho\beta Q^2/A$, which we can write as $\rho\beta QU$, where U is the velocity, and because of the constriction the velocity leaving is greater than the incoming velocity, hence positive momentum has been given to the fluid, and the effect of fluid on the nozzle is indeed a negative force.
- One student asked a good question – what is the difference between the force on a gradually tapering nozzle and one which might converge quite abruptly? The answer is, according to the above theory, no difference, because the shape of the nozzle has not entered our calculations at all. The pressure at 1 is caused only by the difference in velocities between 1 and 2, and the force only by that pressure plus the difference in momenta between 1 and 2.

6. *A nozzle at a wheat terminal is capable of delivering* 0.7 *bags per second into the hold of a ship. (One bag has a mass of* $82\,\mathrm{kg}$ *and a volume of* $110L$*). The nozzle is circular in section with a diameter of* $30\,\mathrm{cm}$*, and is directed downwards and away from the wharf at an angle of* $45°$*. Draw a control volume for the stream of wheat and the hold of the ship. If six such nozzles are operating, estimate the force on the ship and what is its horizontal component tending to move the ship away from the wharf? (Ans:* $375\,\mathrm{N}$*,* $265\,\mathrm{N}$*).*