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A BOUSSINESQ APPROXIMATION FOR OPEN CHANNEL FLOW

John D. Fenton[(1)] and Yebegaeshet T. Zerihun[(2)]

[(1)] Institute for Hydromechanics, University of Karlsruhe, Kaiserstr. 12 - 10.81, 76131 Karlsruhe, Germany
fenton@ifh.uka.de, http://www.ifh.uni-karlsruhe.de/people/fenton/, Tel: +49 721 608 7245

[(2)] Dept of Water, Land and Biodiversity Conservation, GPO Box 2834, Adelaide, SA 5001, Australia
Zerihun.Yebegaeshet@saugov.sa.gov.au, Tel: +61 8 8463 6986, Fax: +61 8 8463 6999

ABSTRACT

The one-dimensional long wave equations are used throughout open channel hydraulics as a simple model of complicated reality. Nevertheless they suffer from an important limitation, that if the flow ever becomes critical, they break down. Boussinesq in 1877 developed a non-hydrostatic approximation that leads to a relatively simple correction with the ability to describe both sub- and super-critical flow and transitions between them. That equation, which might have formed the basis of open-channel hydraulics, has instead been substantially ignored. This paper develops a similar approach to Boussinesq, but for channels of greater generality. When compared with experiment it is found to provide a good model for flows that pass through critical, in flumes, over weirs and embankments, and in undular bores.

Key words: Boussinesq, flume, gradually-varied, long waves, non-hydrostatic, open channel, trans-critical, weir

1 INTRODUCTION

The usual distinguishing characteristic for so-called Boussinesq equations or approximations is the inclusion of non-trivial vertical variation of quantities like velocity and pressure. Such approximations occur throughout geophysical fluid mechanics and coastal engineering. In open channel hydraulics, however, Boussinesq's equations, which are the next level of approximation relative to the long wave or Saint-Venant equations, have remained almost unknown.

Boussinesq's approach to the open channel problem was essentially a classical one dimensional hydraulic one, where the non-hydrostatic pressure effects were modelled approximately without solving the flow problem in three dimensions. He (Boussinesq 1877, described rather more accessibly on p122 *et seq.* of Jaeger 1956) obtained a third-order differential equation for momentum conservation in steady flow in a wide rectangular channel of varying longitudinal bed topography. That equation could perhaps have formed the basis for much of open channel hydraulics, as it is not much longer than the familiar gradually-varied flow equation, while the third derivative term has the power to ensure uniform validity of the solution as the flow passes through what would otherwise be critical. That derivative might have proved a disincentive to hydraulicians, but a simpler possible explanation for the apparent reluctance to adopt Boussinesq's equation is that Jaeger described it in almost pejorative terms, and subsequent writers of monographs have all but ignored it.

Jaeger (1956, Chapter IV, §2, equation 4) presented Boussinesq's equation for steady flow written in terms of an independent curvilinear variable s along the possibly-curved channel bottom, and then only for a wide channel. It is consistent within the approximations of such a long-wave approach to replace the curvilinear co-ordinate s with a cartesian co-ordinate x, so that here we write it as

$$\frac{q^2}{3gh}\frac{d^3h}{dx^3} + \left(1-\beta F^2\right)\frac{dh}{dx} = \frac{q^2}{2gh}S_0'' + S_0 - \frac{q^2}{k^2}, \tag{1}$$

where h is depth relative to the bed, q and k are the discharge and conveyance, both per unit width, bed slope is S_0 and $S_0'' = d^2S_0/dx^2$, β is a Boussinesq coefficient, $\beta > 1$, that allows for effects of turbulence (Fenton 2005) and non-uniformity of velocity distribution; and $F^2 = q^2/gh^3$ is the square of the Froude number, in which g is gravitational acceleration. For uniform flow over a constant slope, the equation has the solution $q = k\sqrt{S_0}$, such that k is given by a resistance law such as Weisbach, Chézy, or Gauckler-Manning-Strickler. Equation (1) can be compared with the well-known gradually-varied flow equation obtained from the long-wave equations based on a hydrostatic pressure assumption:

$$\frac{dh}{dx} = \frac{S_0 - q^2/k^2}{1 - \beta F^2}, \tag{2}$$

and it can be seen that the only differences are two extra terms of the same nature in equation (1), namely the third derivative of the dependent variable h on the left and S_0'', the negative of the third derivative of the bed elevation, on the right. The possible difficulty of a non-hydrostatic pressure distribution has been able to be incorporated without much additional complication.

It is well-known that the gradually varied flow equation (2) is unable to describe flow near critical ($\beta F^2 \to 1$), or the passage of flow between sub- and super-critical states, when $\beta F^2 = 1$ at some point, as the equation predicts an infinite surface slope there. Traditional sources (e.g. Jaeger 1956, p130; Chow 1959, p237; and Henderson 1966, p106) claim that the equation can describe the transitional flow over a mild slope to flow over a steep slope, but the argument depends on both numerator and denominator of the equation, each different functions of x and h, going to zero at the same point in (x, h) space, a rare and fortuitous occurrence.

Since Boussinesq's contribution there have been several theoretical studies, most of which have incorporated some form of two-dimensional solution, usually by assuming potential flow. A description of those approaches has been given in Zerihun (2004, Chapter 2), but they will not be considered further here, as this work will concentrate on hydraulic or one-dimensional approaches.

Dressler (1978) introduced curvilinear coordinates based on the channel bottom, such that the pressure distribution contained a centrifugal correction to the hydrostatic pressure, and obtained equations equivalent to the long-wave equations. However they also show singular behaviour at critical flow and are unable to describe the transition from sub- to super-critical flow. Steffler & Jin (1993) developed a set of vertically averaged and moment equations, in which they assumed a linear longitudinal velocity distribution, and quadratic vertical and pressure distributions. The equations are long.

Fenton (1996) used an hydraulic approach, unknowingly similar to Boussinesq, but assuming a constant centrifugal acceleration term across a vertical section to give steady flow equations that included third-order derivatives similar to Boussinesq. This was subsequently generalised by Zerihun (2004) to allow for a linear variation of centrifugal acceleration across the flow, and a more satisfactory numerical method was developed, in which the third-order differential equation was solved as a boundary value problem, rather than an initial value problem with parasitic solutions as found in Fenton's original work. Both equations have been compared (Zerihun 2004, Zerihun & Fenton 2006) with a number of practical problems of weirs, contractions, flumes and overfalls in which flow passes through critical. Both gave good agreement with all experimental results. Although the underlying structure of the equations used was simple and similar to that of Boussinesq's equation, they had relatively long and complicated coefficients. It is the aim of this paper to develop a simpler approach, emulating that of Boussinesq, but using cartesian co-ordinates and applicable also to unsteady flows and to channels of arbitrary section. The equations so obtained, and special cases, give agreement with experiment as good as the earlier approaches, and are to be preferred to them. They can be used for a variety of problems where flow may pass through critical such as in flumes and over broad-crested weirs.

2 THEORY

2.1 DERIVATION

The momentum equation for open channel flow can be written for a straight channel in surprisingly simple terms and with few approximations:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta \frac{Q^2}{A}\right) + \frac{1}{\rho}\int_A \frac{\partial p}{\partial x} dA = \beta_i i\, u_i - \frac{\bar{\tau} P}{\rho}, \tag{3}$$

where Q is discharge, A is area, β is a Boussinesq momentum coefficient, p is pressure, ρ is fluid density, $\beta_i i\, u_i$ is the contribution of an inflow of i volume rate per unit length, with streamwise velocity component u_i, $\bar{\tau} P$ is the integral of shear stress τ around the boundary, with wetted perimeter P. The x cartesian co-ordinate is horizontally along the channel.

The friction term can be approximated by the Weisbach formula, and assuming that the mean fluid velocity parallel to the boundary is $Q/\left(A\cos\theta\right)$, where θ is the slope angle of the bed, the last term in equation (3) can be written

$$\frac{\bar{\tau} P}{\rho} = \frac{\bar{\lambda} P}{8} \frac{Q\left|Q\right|}{\cos^2\theta A^2}, \tag{4}$$

where $\bar{\lambda} P$ is the integral of the Weisbach coefficient λ around the boundary, such that $\bar{\lambda}$ is the mean value of the coefficient. The only major problem in using equation (3) is the pressure term $\int_A \partial p/\partial x\, dA$. The conventional approximation in hydraulic engineering is that the pressure is given by the equivalent static head of fluid above each point. Here we attempt to go further by allowing for vertical acceleration effects on the fluid pressure, while retaining a traditional hydraulics approach, where the problem will be modelled as adequately as necessary, but as simply as possible.

In Fenton (1996) a centrifugal approach in terms of radius of curvature of particle paths was adopted, which led to a rather complicated equation. Here, a simpler approach is made, using the vertical component of Euler's equation of motion (*e.g.* §4.3 of White 2003):

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g - \frac{Dw}{Dt}, \tag{5}$$

where w is the vertical velocity of a fluid particle, such that Dw/Dt is its vertical acceleration

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}. \tag{6}$$

Here, it is observed that in most hydraulics problems the rates of change of quantities with time are much slower than the apparent acceleration of a fluid particle following a curved path. Also, motion across the channel plays a less-important role than the other two directions. The most important component in equation (6) is the term $u\, \partial w/\partial x$, giving the vertical acceleration as a particle of horizontal velocity u follows a path which is curved in the vertical such that its vertical velocity is changing. Let the particle path be specified by $z = \zeta$, then the vertical velocity w is approximated by

$$w \approx \frac{dx}{dt}\frac{d\zeta}{dx} = u\zeta_x, \quad \text{and the apparent acceleration} \quad \frac{Dw}{Dt} \approx \frac{du}{dt}\zeta_x + u^2\zeta_{xx}.$$

In the latter it is assumed that the first term, the product of two derivatives, will be relatively small, such that the dominant contribution is that due to the second derivative term. In the spirit of conventional hydraulics this contribution to the pressure gradient is approximated as a constant over the section such that equation (5) is approximated by

$$\frac{1}{\rho}\frac{\partial p}{\partial z} \approx -g - \left(\frac{Q}{A}\right)^2 \bar{\zeta}_{xx}, \tag{7}$$

where $\bar{\zeta}$ can be thought of as the weighted mean elevation of all particle paths over a cross-section. Integrating this equation in the vertical between a point z and the free surface η on which pressure is zero gives an expression for the pressure throughout the water:

$$\frac{p}{\rho} = \left(g + \frac{Q^2}{A^2}\bar{\zeta}_{xx}\right)(\eta - z), \tag{8}$$

where it has been assumed that the free surface is horizontal across the section, such that the pressure is not a function of y. In this context $z = \bar{\zeta}(x)$ can be thought of as defining an elevation, variable along the channel, that approximates the mean elevation of the whole cross-section weighted with respect to higher-velocity fluid.

Differentiating equation (8) with respect to x:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \left(g + \frac{Q^2}{A^2}\bar{\zeta}_{xx}\right)\frac{\partial \eta}{\partial x} + (\eta - z)\frac{\partial}{\partial x}\left(\frac{Q^2}{A^2}\bar{\zeta}_{xx}\right),$$

and integrating across the channel gives

$$\begin{aligned}\frac{1}{\rho}\int\limits_A \frac{\partial p}{\partial x} dA &= \left(g + \frac{Q^2}{A^2}\bar{\zeta}_{xx}\right)\frac{\partial \eta}{\partial x}\int\limits_A dA + \frac{\partial}{\partial x}\left(\frac{Q^2}{A^2}\bar{\zeta}_{xx}\right)\int\limits_A (\eta - z)\, dA \\ &= \left(gA + \frac{Q^2}{A}\bar{\zeta}_{xx}\right)\frac{\partial \eta}{\partial x} + A\bar{D}\frac{\partial}{\partial x}\left(\frac{Q^2}{A^2}\bar{\zeta}_{xx}\right),\end{aligned} \tag{9}$$

where the first integral $\int_A dA = A$, the area, and the second integral $\int_A (\eta - z)\, dA$ is the first moment of area of the cross-section about an axis transverse to the flow across the free surface, which we have written as $A\bar{D}$, where $\bar{D}$ is the depth of the centroid of the section below the free surface. Substituting expression (9) into the momentum equation (3) and using equation (4) for the resistance term, the integral momentum equation becomes:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right) + \left(gA + \frac{Q^2}{A}\bar{\zeta}_{xx}\right)\frac{\partial \eta}{\partial x} + A\bar{D}\frac{\partial}{\partial x}\left(\frac{Q^2}{A^2}\bar{\zeta}_{xx}\right) = \beta_i i\, u_i - gA\frac{Q\,|Q|}{K^2}, \tag{10}$$

where the conveyance K has been introduced as a convenient shorthand,

$$K = A\cos\theta\sqrt{\frac{A}{P}\frac{8g}{\bar{\lambda}}} = CA\cos\theta\sqrt{\frac{A}{P}}, \tag{11}$$

where $C = \sqrt{8g/\bar{\lambda}}$ is the Chézy coefficient.

Before expanding the derivative terms in equation (10), it is necessary to express $\partial A/\partial x$ in terms of the section properties. The area A is defined by the integral of the depth in across the channel

$$A = \int\limits_{Y_\mathrm{R}}^{Y_\mathrm{L}} (\eta - Z)\, dy,$$

where the stream bed is defined by $z = Z(x, y)$, and $y = Y_\mathrm{R}$ and $y = Y_\mathrm{L}$ define the right and left waterlines, which are functions of x and the local surface elevation $\eta(x, t)$. Leibniz' rule for the derivative of an integral gives

$$\frac{\partial A}{\partial x} = B\frac{\partial \eta}{\partial x} - \int\limits_{Y_{\mathrm{R}}}^{Y_{\mathrm{L}}} \frac{\partial Z}{\partial x}\, dy + (\eta - Z)_{\mathrm{L}} \frac{\partial Y_{\mathrm{L}}}{\partial x} - (\eta - Z)_{\mathrm{R}} \frac{\partial Y_{\mathrm{R}}}{\partial x}. \tag{12}$$

As $\partial Z/\partial x$ is the local downstream slope of the stream bed, the notation

$$\bar{S} = -\frac{1}{B}\int\limits_{Y_{\mathrm{R}}}^{Y_{\mathrm{L}}} \frac{\partial Z}{\partial x}\, dy \tag{13}$$

for the integral is suggested, denoting the mean stream bed slope at a section. It has a negative sign such that in the usual situation where the bed slopes downwards in the direction of x, $\bar{S}$ will be positive. The last two terms in equation (12) are denoted by

$$A_x^{\mathrm{V}} = (\eta - Z)_{\mathrm{L}} \frac{\partial Y_{\mathrm{L}}}{\partial x} - (\eta - Z)_{\mathrm{R}} \frac{\partial Y_{\mathrm{R}}}{\partial x}, \tag{14}$$

the notation V suggesting the contribution to $\partial A/\partial x$ which is non-zero only in the case of *vertical* converging or diverging side walls, such that $(\eta - Z)_{\mathrm{L}}\, \partial Y_{\mathrm{L}}/\partial x$ and $(\eta - Z)_{\mathrm{R}}\, \partial Y_{\mathrm{R}}/\partial x$ are non-zero. In the usual situation where the banks of the stream are not vertical, the depth $\eta - Z$ is zero at both banks, and $A_x^{\mathrm{V}} = 0$.

Using equations (13) and (14), equation (12) can be written

$$\frac{\partial A}{\partial x} = B\left(\frac{\partial \eta}{\partial x} + \bar{S}\right) + A_x^{\mathrm{V}}, \tag{15}$$

In the case of uniform canals $\bar{S}$ is well-defined as simply the bed slope. Expanding equation (10) using this equation gives

$$\begin{aligned}\frac{\partial Q}{\partial t} + 2\left(\beta + \bar{D}\bar{\zeta}_{xx}\right)\frac{Q}{A}\frac{\partial Q}{\partial x} + \left(gA + \frac{Q^2 B}{A^2}\left(\left(\frac{A}{B} - 2\bar{D}\right)\bar{\zeta}_{xx} - \beta\right)\right)\frac{\partial \eta}{\partial x} + \frac{Q^2\bar{D}}{A}\bar{\zeta}_{xxx} \\ = \beta_i\, i\, u_i - \frac{Q^2}{A}\beta' + \frac{Q^2}{A^2}\left(\beta + 2\bar{D}\bar{\zeta}_{xx}\right)\left(B\bar{S} + A_x^{\mathrm{V}}\right) - gA\frac{Q\,|Q|}{K^2}.\end{aligned} \tag{16}$$

In three places in this equation, a β coefficient is combined with a term in $\bar{\zeta}_{xx}$, such as $\beta + \bar{D}\bar{\zeta}_{xx}$. In each case that second derivative term is likely to be small (of the order of depth divided by the radius of curvature of the streamlines), and as β is not well-known anyway, it is proposed to neglect such second derivative terms. This will not change the overall behaviour of the equation and should have a small effect on solutions. The term in $\beta' = d\beta/dx$ on the right will also be neglected. The resulting equation is

$$\frac{\partial Q}{\partial t} + 2\beta\frac{Q}{A}\frac{\partial Q}{\partial x} + \left(gA - \beta\frac{Q^2 B}{A^2}\right)\frac{\partial \eta}{\partial x} + \frac{Q^2\bar{D}}{A}\bar{\zeta}_{xxx} = \beta_i\, i\, u_i + \beta\frac{Q^2}{A^2}\left(B\bar{S} + A_x^{\mathrm{V}}\right) - gA\frac{Q\,|Q|}{K^2}. \tag{17}$$

Now we let the elevation of the mean streamline at a section be a weighted mean of the transverse mean bed elevation and the free surface:

$$\bar{\zeta}_{xxx} = \gamma\eta_{xxx} + (1 - \gamma)\,\bar{Z}''', \tag{18}$$

the magnitude of γ and $1 - \gamma$ reflecting the relative contribution of flow near the surface and near the bed. Substituting into (17), and recognising that $\bar{Z}''' = -\bar{S}''$,

$$\frac{\partial Q}{\partial t} + 2\beta\frac{Q}{A}\frac{\partial Q}{\partial x} + \left(gA - \beta\frac{Q^2B}{A^2}\right)\frac{\partial \eta}{\partial x} + \gamma\frac{Q^2\bar{D}}{A}\frac{\partial^3\eta}{\partial x^3} = \\ \beta_i\, i\, u_i + (1-\gamma)\frac{Q^2\bar{D}}{A}\bar{S}'' + \beta\frac{Q^2}{A^2}\left(B\bar{S} + A_x^{\mathrm{v}}\right) - gA\frac{Q\,|Q|}{K^2}. \quad (19)$$

This is the Boussinesq equation that has been sought, the unsteady equation for an arbitrary channel section, which has required only two higher-derivative terms more than the Saint-Venant long wave equation. It is now used for some other cases.

2.2 EQUATION IN TERMS OF A LOCAL DEPTH-LIKE VARIABLE

In the case of any channel that is not rectangular, the concept of "depth" is ambiguous, especially for natural channels. However it is often convenient to express local quantities such as conveyance in terms of the local height above the bed or something similar. Here we introduce the depth-like quantity h that is the height of the free surface above a reference axis that can be chosen arbitrarily, a variable height $z_0(x)$ above the x axis. For a channel with a horizontally-flat bottom, this axis would be chosen to coincide with the bottom, but for other channels it is arbitrary, and h is not strictly depth. We have

$$\eta = h + z_0, \quad \text{and} \quad \frac{\partial \eta}{\partial x} = \frac{\partial h}{\partial x} + \frac{\partial z_0}{\partial x} = \frac{\partial h}{\partial x} - S_0,$$

where $S_0 = -\partial z_0/\partial x$, the slope of the reference axis, which in the case of a horizontally-flat bed, is the longitudinal bed slope. Substituting into equation (19)

$$\frac{\partial Q}{\partial t} + 2\beta\frac{Q}{A}\frac{\partial Q}{\partial x} + \left(gA - \beta\frac{Q^2B}{A^2}\right)\frac{\partial h}{\partial x} + \gamma\frac{Q^2\bar{D}}{A}\frac{\partial^3 h}{\partial x^3} \;=\; \beta_i\, i\, u_i + \frac{Q^2\bar{D}}{A}\left((1-\gamma)\bar{S}'' + \gamma S_0''\right) \\ + gAS_0 + \beta\frac{Q^2}{A^2}\left(B\bar{S} - BS_0 + A_x^{\mathrm{v}}\right) - gA\frac{Q\,|Q|}{K^2}. \quad (20)$$

Equations (19) and (20) are the same equation; the former is written with surface elevation η as dependent variable and the latter with the depth-like quantity h.

2.3 HYDRAULICALLY-WIDE CHANNEL

For the case of a wide channel with a horizontal bottom, and width B, if the discharge is q per unit width, then $Q = Bq$, $A = Bh$, $\bar{D} = h/2$, $P = B$, $A_x^{\mathrm{v}} = 0$, $K = Bk$, and using $\bar{S} = S_0$, equations (19) and (20) give, for the two different dependent variables η and h,

$$\begin{aligned} \frac{\partial q}{\partial t} + 2\beta\frac{q}{h}\frac{\partial q}{\partial x} + \left(gh - \beta\frac{q^2}{h^2}\right)\frac{\partial \eta}{\partial x} + \gamma\frac{q^2}{2}\frac{\partial^3\eta}{\partial x^3} &= \frac{\beta_i\, i\, u_i}{B} + (1-\gamma)\frac{q^2}{2}S_0'' + \beta\frac{q^2}{h^2}S_0 - gh\frac{q\,|q|}{k^2}, \\ \frac{\partial q}{\partial t} + 2\beta\frac{q}{h}\frac{\partial q}{\partial x} + \left(gh - \beta\frac{q^2}{h^2}\right)\frac{\partial h}{\partial x} + \gamma\frac{q^2}{2}\frac{\partial^3 h}{\partial x^3} &= \frac{\beta_i\, i\, u_i}{B} + \frac{q^2}{2}S_0'' + ghS_0 - gh\frac{q\,|q|}{k^2}. \quad (21) \end{aligned}$$

2.4 STEADY FLOW

In steady flow, $\partial/\partial t = 0$, the mass-conservation equation becomes $\partial Q/\partial x = i$, and as there is no unsteady reversal of flow possible, $Q\,|Q| = Q^2$. Substituting these into equations (19) and (20) and dividing through by Q^2B/A^2, we have the two equivalent third-order ordinary differential equations in terms of η and h:

$$\gamma\frac{A\bar{D}}{B}\frac{d^3\eta}{dx^3} + \left(\frac{1}{F^2} - \beta\right)\frac{d\eta}{dx} = \left(\beta_i\, u_i - 2\beta\frac{Q}{A}\right)\frac{iA^2}{Q^2B} + (1-\gamma)\frac{A\bar{D}}{B}\bar{S}'' + \beta\left(\bar{S} + \frac{A_x^{\mathrm{v}}}{B}\right) - \frac{gA^3}{BK^2}, \quad (22)$$

$$\gamma \frac{A\bar{D}}{B}\frac{d^3h}{dx^3} + \left(\frac{1}{F^2} - \beta\right)\frac{\partial h}{\partial x} = \left(\beta_i\, u_i - 2\beta\frac{Q}{A}\right)\frac{iA^2}{Q^2B} + \frac{A\bar{D}}{B}\left((1-\gamma)\bar{S}'' + \gamma S_0''\right) + \frac{S_0}{F^2} + \beta\left(\bar{S} - S_0 + \frac{A_x^{\text{v}}}{B}\right) - \frac{gA^3}{BK^2} \tag{23}$$

where $F^2 = Q^2B/gA^3$, the square of the Froude number.

2.5 STEADY FLOW IN A WIDE CHANNEL – COMPARISON WITH BOUSSINESQ

For the combination of both special cases of equations (21) and (23), for steady flow in a wide channel, the resulting equation in terms of h becomes

$$\gamma\frac{h^2}{2}\frac{d^3h}{dx^3} + \left(\frac{1}{F^2} - \beta\right)\frac{dh}{dx} = \frac{h^2}{2}S_0'' + \frac{S_0}{F^2} - \frac{gh^3}{k^2}. \tag{24}$$

This can now be compared with Boussinesq's equation (1) by dividing that equation through by $F^2 = q^2/gh^3$ to give

$$\frac{h^2}{3}\frac{d^3h}{dx^3} + \left(\frac{1}{F^2} - \beta\right)\frac{dh}{dx} = \frac{h^2}{2}S_0'' + \frac{S_0}{F^2} - \frac{gh^3}{k^2}, \tag{25}$$

The only difference is in the coefficient in front of the third derivative term, which is $\gamma/2$ in the present equation (24) and $1/3$ in the Boussinesq equation (25). Boussinesq's derivation incorporated a linear variation of the vertical acceleration across the depth, whereas the present work assumed it constant, but that it was determined by a weighted combination of curvature on the bed and at the surface. The two would agree were $\gamma = 2/3$, a quite reasonable value, which we will see is born out by comparison with experiment.

3 COMPARISON WITH EXPERIMENT

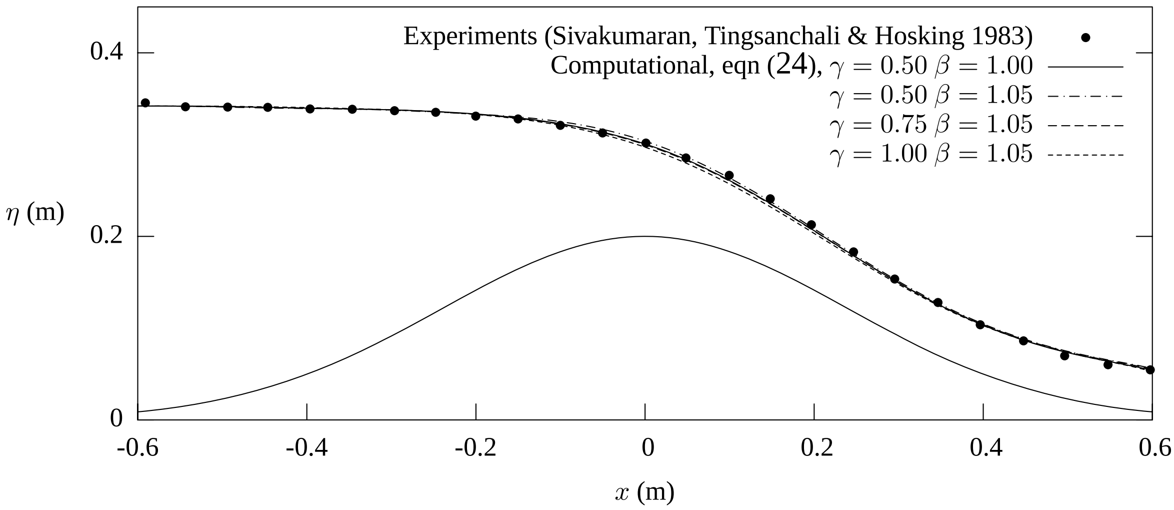

Figure 1. Steady flow over Gaussian hump, computational results for surface profile compared with experimental values of Sivakumaran et al. (1983)

The performance of different Boussinesq equations was compared and tested against several sets of experimental results. Figure 1 shows a comparison with the results from figure 7(a) of Sivakumaran et al. (1983) for flow over a Gaussian hump, passing from sub-critical flow through critical, to super-critical. In particular the performance of equation (24) for different values of the Boussinesq coefficient β and the weighting parameter γ for the free surface are compared. It can be seen that using $\gamma = 1$, ignoring the bed curvature, was slightly less accurate than other results. A close examination of them showed that, using a physically-reasonable value of $\beta = 1.05$, the closest approximation to experiments was obtained using $\gamma = 0.75$. The actual value is not all that important, for it can be seen that all results succeeded in simulating the experiment well, as distinct from results from the gradually-varied flow equation which, as expected, broke down when the flow approached critical.

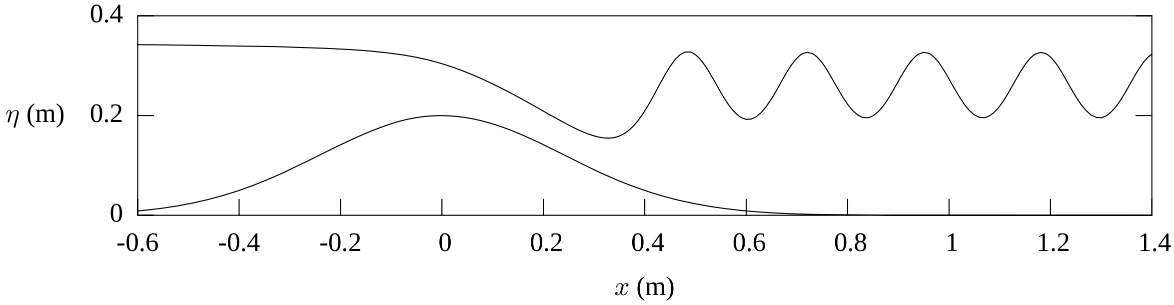

Figure 2. Computational results for the same flow as in Figure 1, but over a slightly lower hump, producing an undular hydraulic jump

In that simulation the relatively simple approach of solving the third-order differential equation with three initial conditions was used, as done by Fenton (1996), and using a fourth-order Runge-Kutta solution method to advance the solution. The method ultimately fails for super-critical flow, as shown by Fenton, that without a downstream boundary condition the differential equation possesses parasitic solutions that diverge to infinity. This slightly questionable procedure, of not specifying a downstream boundary condition, did, however, enable the simulation of an undular hydraulic jump, as shown in Figure 2, when computations were robust and accurate. It can be shown that the third derivative in the Boussinesq equations enables periodic or quasi-periodic solutions such as an undular jump, often observed in nature. The gradually-varied flow equation, as a first-order equation, allows no such solutions.

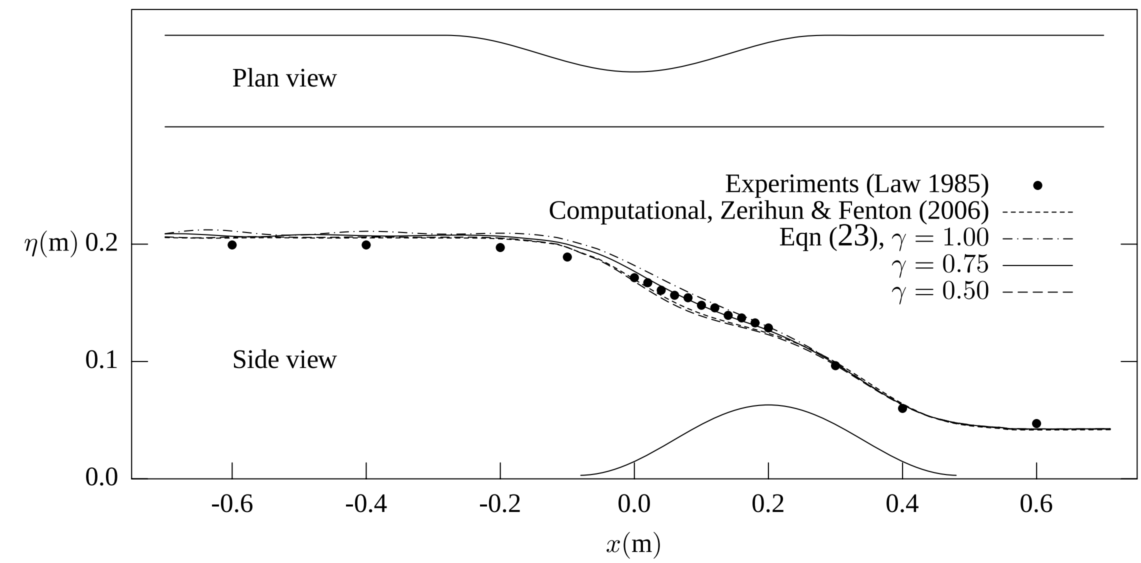

Figure 3. Flow through contraction and over hump (Law 1985)

Figure 3 shows the results of a more general case, that of flow over a hump, but where the side walls contract also, as described and measured by Law (1985). For this case, one of the rather more general equations (23) was used, allowing for horizontal variation of the side walls. It can be seen that the results from that equation provide good agreement with experiment, and that for a value of $\gamma = 0.75$, if anything they are more accurate than the results from the more complicated equation used by Zerihun & Fenton (2006).

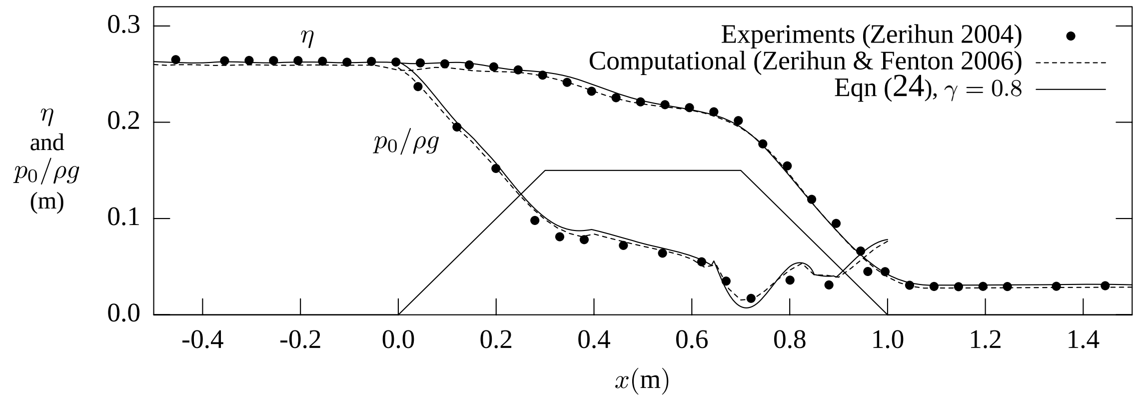

Figure 4. Flow over a broad-crested trapezoidal weir, showing surface elevation η and pressure head on the bed $p_0/\rho g$

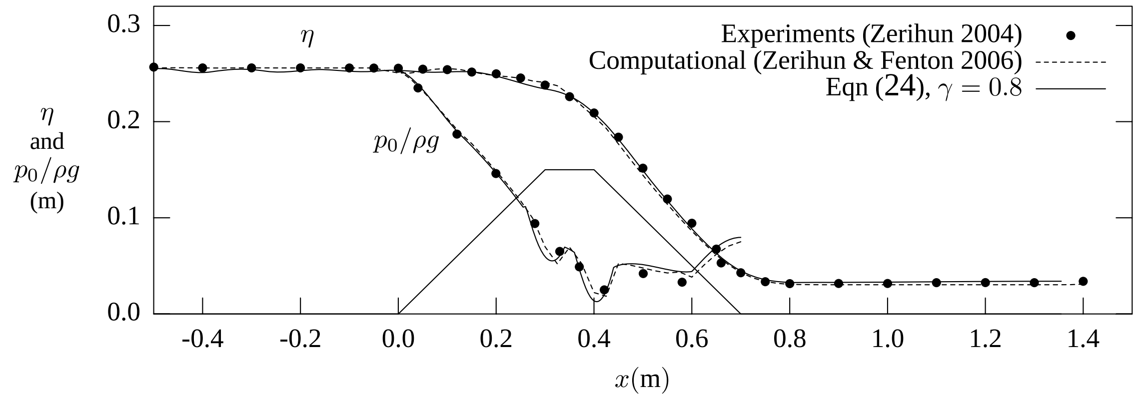

Figure 5. Flow over a short-crested trapezoidal weir

Results for trapezoidal embankments or weirs are shown in Figures 4 and 5. Plotted are the experimental results of Zerihun (2004), both for the elevation of the free surface η and the pressure head on the bed $p_0/\rho g$ obtained from equation (8) and the approximation for the second derivative $\bar{\zeta}_{xx} = \gamma\eta_{xx} + (1-\gamma)\,\bar{Z}''$, similar to equation (18):

$$\frac{p_0}{\rho g} = \left(1 + \frac{Q^2}{gA^2}\left(\gamma\eta_{xx} + (1-\gamma)\,\bar{Z}''\right)\right)(\eta - Z)\,. \tag{26}$$

There is little difference between the use of the present simpler equation (24) and the more complicated equation of Fenton (1996), although the latter is slightly more accurate. Both sets of computations describe the free surface and the bed pressure well.

4 CONCLUSIONS

A one-dimensional channel model has been developed, similar to that of Boussinesq, but for general channel cross-sections. It includes the effect of curvature of bed and free surface, in making the pressure distribution non-hydrostatic, and overcomes the problem that the long wave equations are singular when flow is critical. The equation has been shown, when compared with experiment, to be a good model for flows that pass through critical, in flumes, over weirs and embankments, and in undular bores. It is, in principle, little more complicated than the conventional long-wave equation, although the presence of a third derivative makes it more difficult to solve numerically. It is simpler than the model presented in Fenton (1996) and described in Zerihun & Fenton (2006), and is to be preferred to that formulation.

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