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A ninth-order solution for the solitary wave

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Several solutions for the solitary wave have been attempted since the work of Boussinesq in 1871. Of the approximate solutions, most have obtained series expansions in terms of wave amplitude, these being taken as far as the third order by Grimshaw (1971). Exact integral equations for the surface profile have been obtained by Milne-Thomson (1964, 1968) and Byatt-Smith (1970), and these have been solved numerically. In the present work an exact operator equation is developed for the surface profile of steady water waves. For the case of a solitary wave, a form of solution is assumed and coefficients are obtained numerically by computer to give a ninth-order solution. This gives results which agree closely with exact numerical results for the surface profile, where these are available. The ninth-order solution, together with convergence improvement techniques, is used to obtain an amplitude of 0·85 for the solitary wave of greatest height and to obtain refined approximations to physical quantities associated with the solitary wave, including the surface profile, speed of the wave and the drift of fluid particles.

1. Introduction

The steady finite amplitude solitary wave was first reported by Russell (1844), who made experimental measurements and gave an empirical relationship for the wave speed, which was later obtained by Boussinesq (1871) and Rayleigh (1876) as part of an overall approximate solution. Since then there have been several attempts to improve upon this solution, the first being by McCowan (1891, 1894), who also obtained an estimate of the limiting height of the wave. A theory of steady finite amplitude waves was given by Korteweg & de Vries (1895), who recognized the nature of the approximations involved, named the waves 'cnoidal' and showed that the solitary wave was a particular limiting case, that of infinite wavelength but having a finite effective length.

Friedrichs (1948) produced a systematic approximation procedure using the equations of motion; this approach was later to be used by Laitone (1960) for a second-order solution of cnoidal and solitary waves and by Grimshaw (1971), who obtained a third-order solitary wave solution. Such a solution had previously been obtained by Hunt (1955); this, however, did not have a uniformly convergent expansion in the expression for the profile; higher order terms dominated the solution far from the crest.

In 1954, Benjamin & Lighthill clarified the subject by showing the relevance of physical flow parameters and how a systematic approximation procedure based on a stream function expansion could be used. Hunt (1955) obtained the solution mentioned above by extending a method of Levi-Civita. Finally, another approximate solution was obtained by Long (1956), who gave a fifth-order expression for the wave speed.

All the approaches described above are approximations; an exact integral equation for the solitary wave was obtained by Milne-Thomson (1964, 1968). A more general study was made by Byatt-Smith (1970), who obtained an exact integro-differential equation for steady surface waves and produced numerical solutions for the solitary wave. Strelkoff (1971) used the same approach and obtained 0·85 for the amplitude of the solitary wave of greatest height. Although these methods were exact, explicit solutions could not be obtained, and the equations for the surface profile had to be solved numerically. No other physical quantities were calculated.

The present method follows the procedure of Benjamin & Lighthill, carrying their work to a higher order of accuracy. An exact operator equation, which is a differential equation of infinite order, is produced and this is truncated to yield a differential equation for the free surface of cnoidal waves. A systematic expansion procedure is introduced for the particular case of the solitary wave; from this a uniformly valid third-order solution is obtained analytically. Using a computer to expand and manipulate the operators, a ninth-order solitary wave solution is obtained, of which the first three terms agree with the analytic third-order solution.

This ninth-order solution, when transformed using convergence improvement techniques, is seen to give accurate results for solitary waves of finite amplitude and gives expressions for other physical quantities of the wave, namely fluid velocities and drift.

2. Exact operator equation

We consider steady two-dimensional flow, in a plane (x, y), of a homogeneous irrotational incompressible fluid over a horizontal bottom, with gravity acting in the $-y$ direction. The wave train set up is stationary in the x, y plane, with fluid velocity components u and v respectively; the co-ordinate origin is at a point on the rigid bottom with x in the direction of flow.

Three physical parameters can be defined which are characteristic of the wave train and which completely define it: the volume flow rate per unit span Q, the energy per unit mass R, and the force plus momentum flux divided by density per unit span, S. In a wave train without friction or other losses, all three are constant. By definition

$$Q = \int_0^\eta u\,dy, \tag{1}$$

$$R = p/\rho + gy + \tfrac{1}{2}(u^2+v^2), \tag{2}$$

$$S = \int_0^\eta (p/\rho + u^2)\,dy, \tag{3}$$

where $p(x, y)$ is the pressure, ρ is the fluid density and g is the gravitational acceleration. Substituting (2) into (3) and integrating, we have

$$S-R\eta+\tfrac{1}{2}g\eta^2 = \tfrac{1}{2}\int_0^\eta (u^2-v^2)\,dy. \tag{4}$$

Now we have two equations, (1) and (4), connecting the three invariants Q, R and S with the surface elevation η and the velocity components u and v. The flow is irrotational and incompressible, hence a complex function w exists which is analytic in z and is defined by

$$w(z) = \phi+i\psi, \tag{5a}$$

where

$$z = x+iy, \tag{5b}$$

$$\begin{aligned}\frac{dw}{dz} &= \frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}\\ &= \frac{\partial\psi}{\partial y}-i\frac{\partial\phi}{\partial y}\\ &= u-iv.\end{aligned} \tag{5c}$$

We can rewrite (1) and (4) in terms of the complex variables defined in (5):

$$Q = \mathscr{I}\int_0^{i\eta}\frac{dw}{dz}.dz, \tag{6a}$$

$$S-R\eta+\tfrac{1}{2}g\eta^2 = \tfrac{1}{2}\mathscr{I}\int_0^{i\eta}\left(\frac{dw}{dz}\right)^2.dz. \tag{6b}$$

Equation (6a) follows by definition of w; (6b) is the equivalent of the Blasius theorem for the force on a body immersed in an incompressible irrotational fluid; the term on the right-hand side is the contribution to the flow force of the inertia effects.

Now we can assume a definite form for w such that the following two conditions are satisfied. (i) $w(z)$ is analytic. (ii) The velocity is horizontal on the bottom, i.e. for z real, dw/dz is real. Such a function is given by

$$\frac{dw}{dz} = e^{iyD}u(x,0), \tag{7}$$

where we have operator

$$e^{iyD} \equiv 1-\frac{y^2}{2!}\frac{d^2}{dx^2}+\frac{y^4}{4!}\frac{d^4}{dx^4}+\ldots+i\left(y\frac{d}{dx}-\frac{y^3}{3!}\frac{d^3}{dx^3}+\ldots\right)$$

and an operand $u(x, 0)$ which is a function of x alone and which represents the velocity along the botton. Substituting (7) and the similar relationship between $(dw/dz)^2$ and $u^2(x, 0)$ into (6), we have a pair of nonlinear operator equations in $u(x, 0)$ and $\eta(x)$:

$$Q = [\sin\eta D]\,[Iu(x,0)], \tag{8a}$$

$$S-R\eta+\tfrac{1}{2}g\eta^2 = \tfrac{1}{2}[\sin\eta D]\,[Iu^2(x,0)], \tag{8b}$$

where I is an integral operator: $D^nI = D^{n-1}$. We can invert (8a) to give

$$u(x,0) = \left(\frac{\eta D}{\sin\eta D}\right)\left(\frac{Q}{\eta}\right). \tag{9}$$

This is substituted into (8 *b*) to give the symbolic operator equation for the free surface which is an infinite order differential equation in η:

$$S-R\eta+\tfrac{1}{2}g\eta^2 = \tfrac{1}{2}Q^2[\sin\eta D]\,I[(\eta D/\sin\eta D)\,(1/\eta)]^2. \tag{10}$$

3. Expansion of operator equation

We expand the symbolic expressions in (10):

$$\sin\eta D \equiv \eta D-\frac{\eta^3}{3!}D^3+\frac{\eta^5}{5!}D^5-\frac{\eta^7}{7!}D^7+O(\eta^9), \tag{11a}$$

$$\frac{\eta D}{\sin\eta D} \equiv 1+\frac{\eta^2}{3!}D^2\left(1+\frac{\eta^2}{3!}D^2\left(1+\frac{\eta^2}{3!}D^2+\ldots\right)-\frac{\eta^4}{5!}D^4+\ldots\right)$$
$$-\frac{\eta^4}{5!}D^4\left(1+\frac{\eta^2}{3!}D^2+\ldots\right)+\frac{\eta^6}{7!}D^6(1+\ldots)+O(\eta^8). \tag{11b}$$

These expressions are substituted into (10) and the necessary differentiations performed. At this stage we are forced to truncate the infinite operators at the stage of expansion shown in (11), thereby introducing the first approximations into the analysis. We have three dimensions which characterize the wave train: the wave amplitude a, representative depth h and a measure of the horizontal extent of each wave, e, which for periodic waves is the wavelength. After the substitution of (11) into (10) and subsequent truncation, we have the equation for the free surface

$$2S\eta-2R\eta^2+g\eta^3+Q^2(-1+D_1(\eta)+D_2(\eta)+D_3(\eta)) = O(a^3h^3/e^6), \tag{12a}$$

where

$$D_1(\eta) = \tfrac{1}{3}\eta'^2, \tag{12b}$$
$$D_2(\eta) = \tfrac{1}{45}(2\eta^2\eta'\eta'''-\eta^2\eta''^2+2\eta\eta'^2\eta''-12\eta'^4), \tag{12c}$$
$$D_3(\eta) = \tfrac{2}{945}(2\eta^4\eta'\eta^{\mathrm{v}}-2\eta^4\eta''\eta^{\mathrm{iv}}+\eta^4\eta'''^2). \tag{12d}$$

The differential terms have been grouped according to order; the equation represents the contortions which the surface must undergo in order to keep a balance between the velocity on the surface required by energy considerations and that required to maintain irrotationality and incompressibility within the flow. We now have a differential equation, simply an extension of that obtained by Benjamin & Lighthill, which describes the free surface in terms of the three basic parameters of the flow, Q, R and S.

If the free surface is horizontal, all the differential terms disappear and we have a cubic equation in η, giving three possible depths for given Q, R and S. Benjamin & Lighthill have shown that the smallest root is impossible, leaving the two alternative depths known in channel flow theory.

The first differential term $D_1(\eta)$ was included by Benjamin & Lighthill, who deduced from it the equations for cnoidal waves. This order of accuracy requires that $D_2(\eta)$, whose value they gave in a footnote, satisfies $D_2(\eta) \ll 1$, or

$$a^2h^2/e^4 = (a/h)^2\,(h/e)^4 \ll 1.$$

Similarly, if we include the second differential term $D_2(\eta)$ we have the requirement that

$$(a/h)^2\,(h/e)^6 \ll 1$$

and finally, if we include $D_3(\eta)$,

$$(a/h)^3(h/e)^6 \ll 1.$$

For all orders of approximation we can see that the standard of accuracy requires that the amplitude be small and the water shallow.

The criteria set out above apply to all finite amplitude waves; we can show that they can be put in different form for the solitary wave. Ursell (1953) gives, for the solitary wave, $ae^2/h^3 \sim 1$. Substituting this into the above conditions, we have

$$(a/h)^4 \ll 1, \quad (a/h)^5 \ll 1, \quad (a/h)^6 \ll 1$$

respectively, for each order of approximation for the solitary wave.

4. Third-order solitary wave solution

4.1. *Expansion scheme*

Solitary waves are possible when we have a uniform flow for which the Froude number is greater than unity, i.e. $U > (gh)^{\frac{1}{2}}$, where U is the velocity of the uniform stream. In this case,

$$Q = Uh, \tag{13a}$$

$$R = \tfrac{1}{2}U^2 + gh, \tag{13b}$$

$$S = U^2h + \tfrac{1}{2}gh^2 \tag{13c}$$

and we define the Froude number F as

$$F = U/(gh)^{\frac{1}{2}}.$$

Now we non-dimensionalize by defining $\eta_* = \eta/h$, $x_* = x/h$ and suppress the asterisks. We can write (12*a*) for the solitary wave as

$$\eta^3 - (F^2+2)\eta^2 + (2F^2+1)\eta + F^2(-1 + D_1(\eta) + D_2(\eta) + D_3(\eta)) = O(\epsilon^6), \tag{14}$$

where $\epsilon = a/h$. We have a one-parameter family of solutions in F; we use an expansion scheme in terms of the non-dimensional amplitude ϵ and strain the x co-ordinate, following the method of Lighthill (1949), to obtain a uniformly convergent solution. Perturbing about the uniform stream of critical depth, we have

$$\eta(\alpha x) = 1 + \epsilon\eta_1 + \epsilon^2\eta_2 + \epsilon^3\eta_3 + \epsilon^4\eta_4 + \epsilon^5\eta_5 + O(\epsilon^6), \tag{15a}$$

$$F^2 = 1 + \epsilon F_1 + \epsilon^2 F_2 + \epsilon^3 F_3 + \epsilon^4 F_4 + \epsilon^5 F_5 + O(\epsilon^6), \tag{15b}$$

$$\alpha^2 = \epsilon\alpha_1 + \epsilon^2\alpha_2 + \epsilon^3\alpha_3 + \epsilon^4\alpha_4 + \epsilon^5\alpha_5 + O(\epsilon^6). \tag{15c}$$

The expansions (15) are substituted into (14). We find that all terms in ϵ^0, ϵ^1 and ϵ^2 drop out, as do all fourth- and fifth-order constants and functions: F_4, α_4, η_4, F_5, α_5, η_5. Grouping all the remaining terms into coefficients of ϵ^3, ϵ^4 and ϵ^5, each of which must be satisfied individually, we have the following three equations, where primes refer to differentiation with respect to αx:

$$\eta_1^3 - F_1\eta_1^2 + \tfrac{1}{3}\alpha_1\eta_1'^2 = 0, \tag{16a}$$

$$\tfrac{1}{3}(2\alpha_1\eta_1'\eta_2' + \alpha_2\eta_1'^2 + F_1\alpha_1\eta_1'^2) + 3\eta_1^2\eta_2 - 2F_1\eta_1\eta_2 - F_2\eta_1^2 + \tfrac{1}{45}\alpha_1^2(2\eta_1'\eta_1''' - \eta_1''^2) = 0, \tag{16b}$$

$$\tfrac{1}{3}(2\alpha_1\eta_1'\eta_3'+\alpha_1\eta_2'^2+2\alpha_2\eta_1'\eta_2'+\alpha_3\eta_1'^2+2F_1\alpha_1\eta_1'\eta_2'+F_1\alpha_2\eta_1'^2+F_2\alpha_1\eta_1'^2)$$
$$+3\eta_1^2\eta_3+3\eta_1\eta_2^2-F_1\eta_2^2-2F_1\eta_1\eta_3-2F_2\eta_1\eta_2-F_3\eta_1^2$$
$$+\tfrac{1}{45}\alpha_1^2(2\eta_1'\eta_2'''+2\eta_2'\eta_1'''-2\eta_1''\eta_2''+4\eta_1\eta_1'\eta_1'''-2\eta_1\eta_1''^2+2\eta_1'^2\eta_1'')$$
$$+\tfrac{1}{45}(F_1\alpha_1^2+2\alpha_1\alpha_2)(2\eta_1'\eta_1'''-\eta_1''^2)+\tfrac{2}{945}\alpha_1^3(2\eta_1'\eta_1^{\mathrm{v}}-2\eta_1''\eta_1^{\mathrm{iv}}+\eta_1'''^2)=0. \qquad (16c)$$

4.2. *Solution*

We seek solutions of the equations (16) which are all uniformly valid such that no term dominates the solution, especially for large x, so that the ratio of higher order terms to lower order terms remains bounded as $|x|\to\infty$. Equation (16a) can be solved using the definition $\eta = 1+\epsilon$ when $\eta' = 0$. Thus we find that

$$F_1 = 1, \quad \alpha_1 = \tfrac{3}{4}, \qquad (17a, b)$$
$$\eta_1 = \operatorname{sech}^2\alpha(x-x_0), \qquad (17c)$$

where x_0 is an arbitrary constant. We set the co-ordinate origin under the crest such that $x_0 = 0$. The equations (17) represent the Boussinesq solution.

Now we substitute these first-order solutions into (16b). Using the requirement that $\eta_2(0) = 0$, we have

$$F_2 = -\tfrac{1}{20}. \qquad (18a)$$

However, we have a family of solutions for η_2, with α_2 as parameter. In general, these solutions contain terms in $\alpha x \operatorname{sech}^2\alpha x . \tanh\alpha x$, which are symmetric in x but will swamp the other terms as $|x|\to\infty$. As explained above we seek only solutions of uniform validity and, setting the coefficients of the above terms to zero, we can solve to obtain

$$\alpha_2 = -\tfrac{15}{16}, \qquad (18b)$$
$$\eta_2 = -\tfrac{3}{4}\operatorname{sech}^2\alpha x . \tanh^2\alpha x. \qquad (18c)$$

This is the result obtained by Laitone (1960). We can repeat the above procedure and substitute the solution (18) into (16c). By using the same requirements of uniform validity we obtain

$$F_3 = -\tfrac{3}{70}, \quad \alpha_3 = \tfrac{9}{8}, \qquad (19a, b)$$
$$\eta_3 = \tfrac{5}{8}\operatorname{sech}^2\alpha x . \tanh^2\alpha x - \tfrac{101}{80}\operatorname{sech}^4\alpha x . \tanh^2\alpha x. \qquad (19c)$$

This is the third-order solution obtained by Grimshaw (1971). We can write the third-order solution for the solitary wave as

$$\eta = 1+\epsilon s^2-\tfrac{3}{4}\epsilon^2 s^2t^2+\epsilon^3(\tfrac{5}{8}s^2t^2-\tfrac{101}{80}s^4t^2)+O(\epsilon^4), \qquad (20a)$$
$$F^2 = 1+\epsilon-\tfrac{1}{20}\epsilon^2-\tfrac{3}{70}\epsilon^3+O(\epsilon^4), \qquad (20b)$$
$$\alpha = (\tfrac{3}{4}\epsilon)^{\frac{1}{2}}(1-\tfrac{5}{8}\epsilon+\tfrac{71}{128}\epsilon^2)+O(\epsilon^{\frac{7}{2}}), \qquad (20c)$$

where $s = \operatorname{sech}\alpha x$, $t = \tanh\alpha x$. We substitute (20a) back into (9) and (7) to obtain the expressions for the velocity at any point:

$$u/(gh)^{\frac{1}{2}} = 1+\tfrac{1}{2}\epsilon-\tfrac{3}{20}\epsilon^2+\tfrac{3}{56}\epsilon^3-\epsilon s^2+\epsilon^2[-\tfrac{1}{4}s^2+s^4+y^2(\tfrac{3}{2}s^2-\tfrac{9}{4}s^4)]$$
$$+\epsilon^3[\tfrac{19}{40}s^2+\tfrac{1}{5}s^4-\tfrac{6}{5}s^6+y^2(-\tfrac{3}{2}s^2-\tfrac{15}{4}s^4+\tfrac{15}{2}s^6)$$
$$+y^4(-\tfrac{3}{8}s^2+\tfrac{45}{16}s^4-\tfrac{45}{16}s^6)]+O(\epsilon^4), \qquad (20d)$$

$$v/(gh)^{\frac{1}{2}} = (3\epsilon)^{\frac{1}{2}}yt\{-\epsilon s^2+\epsilon^2[\tfrac{3}{8}s^2+2s^4+y^2(\tfrac{1}{2}s^2-\tfrac{3}{2}s^4)]$$
$$+\epsilon^3[\tfrac{49}{640}s^2-\tfrac{17}{20}s^4-\tfrac{18}{5}s^6+y^2(-\tfrac{13}{16}s^2-\tfrac{25}{16}s^4+\tfrac{15}{2}s^6)$$
$$+y^4(-\tfrac{3}{40}s^2+\tfrac{9}{8}s^4-\tfrac{27}{16}s^6)]\}+O(\epsilon^{\frac{9}{2}}). \qquad (20e)$$

These are the expressions obtained by Grimshaw, except for an error of his in the signs of the ϵ^3 component of v. We can substitute these into (2) to obtain an expression for the pressure at any point:

$$\begin{aligned} p/\rho g h = {} & 1-y+\epsilon s^2+\epsilon^2[\tfrac{3}{4}s^2-\tfrac{3}{2}s^4+y^2(-\tfrac{3}{2}s^2+\tfrac{9}{4}s^4)] \\ & +\epsilon^3[-\tfrac{1}{2}s^2-\tfrac{19}{20}s^4+\tfrac{11}{5}s^6+y^2(\tfrac{3}{4}s^2+\tfrac{39}{8}s^4-\tfrac{33}{4}s^6) \\ & +y^4(\tfrac{3}{8}s^2-\tfrac{45}{16}s^4+\tfrac{45}{16}s^6)]+O(\epsilon^4). \end{aligned} \tag{20f}$$

4.3. *Fluid drift due to passage of solitary wave*

Price (1971) has calculated the movement of fluid particles on the channel bottom due to the passage of a solitary wave, correct to the second order. This can be extended to the third order using Grimshaw's results, and at the same time we shall extend the theory to give the drift on any streamline.

In this section we consider the unsteady case of a solitary wave passing through a stationary fluid in a stationary co-ordinate frame (x^*, y^*). The wave propagates in the negative x^* direction at speed U and has all the physical quantities defined above in a co-ordinate frame (x, y) moving with the wave; t is the time taken for a particle to reach (x, y), starting from $x = 0$ at $t = 0$, that is, from under the crest of the wave. We can thus write

$$t = \int_0^x \frac{dx}{u(x, y)},$$

where $u(x, y)$ is the horizontal velocity of a particle in the moving frame (x, y). y is no longer an independent variable, however, but varies with x so as to describe the streamline along which the particle moves. Thus we can write more properly

$$t_p = \int_0^{x_p} \frac{dx}{u(x, \psi_p)},$$

where p refers to a particular fluid particle.

The displacement of the crest in time t is $-Ut$, hence

$$\begin{aligned} x_p^* &= x_p - Ut_p \\ &= -\int_0^{x_p} \left(\frac{U}{u(x, \psi_p)} - 1\right) dx, \end{aligned} \tag{21}$$

where x_p^* is the distance moved by a particle p in fixed co-ordinates, and will be negative because the particle moves upstream. As the wave moves to infinity, x_p^* approaches a limit which is half the total drift because the other half of the drift has already occurred by the time the wave crest reaches $x^* = 0$. Thus we can write, because the wave is symmetrical,

$$\text{total drift} = -\int_{-\infty}^{\infty} \left(\frac{U}{u(x, \psi_p)} - 1\right) dx.$$

We can suppress the negative sign, remembering that drift always occurs in the direction of propagation of the wave, and define $\delta(\psi)$ to be the drift on a streamline ψ due to the passage of a solitary wave:

$$\delta(\psi) = \int_{-\infty}^{\infty} \left(\frac{U}{u(x, \psi)} - 1\right) dx. \tag{22}$$

We now evaluate this drift using the third-order expression for velocity. First, we need to know the elevation of any streamline at any point x; to find this we integrate ($20d$) with respect to y to give ψ, and invert the equation to obtain the elevation of any streamline ψ:

$$\begin{aligned} y(x,\psi) = {} & \psi+\epsilon\psi s^2+\epsilon^2\psi^3(-\tfrac{3}{4}s^2+\tfrac{3}{4}s^4) \\ & +\epsilon^3[(-\tfrac{1}{5}\psi+\tfrac{3}{4}\psi^3+\tfrac{3}{40}\psi^5)s^2+(\tfrac{9}{20}\psi-\tfrac{11}{8}\psi^3-\tfrac{9}{16}\psi^5)s^4 \\ & +(\tfrac{1}{5}\psi+\tfrac{1}{2}\psi^3+\tfrac{9}{16}\psi^5)s^6]+O(\epsilon^4). \end{aligned} \tag{23}$$

For $\psi = 1$, this recovers the equation for the free surface, ($20a$). Now we substitute this into ($20d$) and invert the equation to give

$$\begin{aligned} \delta(\psi) &= \int_{-\infty}^{\infty} \{\epsilon s^2+\epsilon^2[-\tfrac{1}{4}s^2+\psi^2(-\tfrac{3}{2}s^2+\tfrac{9}{4}s^4)] \\ &\qquad +\epsilon^3[-\tfrac{1}{5}s^2-\tfrac{1}{5}s^4+\tfrac{1}{5}s^6+\psi^2(\tfrac{21}{8}s^2-\tfrac{99}{16}s^4+\tfrac{69}{16}s^6)]\}\,dx \\ &= 2(\tfrac{4}{3}\epsilon)^{\frac{1}{2}}(1+\tfrac{3}{8}\epsilon+\epsilon^2(-\tfrac{5251}{9600}+\tfrac{4}{5}\psi^2))+O(\epsilon^{\frac{7}{2}}). \end{aligned} \tag{24}$$

This gives Price's second-order result for the bottom drift and also shows that to second order the drift is constant throughout the fluid. The third-order term, however, gives a large correction to Price's result and brings it nearer to his alternative calculation based on Lenau's result for the wave of greatest height. This third-order term has a coefficient of $-5251/9600$ for the bottom drift, while for the surface it has a value of $+2429/9600$. Therefore, surface drift is considerably larger than the bottom drift, as we might expect. The total volume of drift V can be obtained by integrating (24):

$$\begin{aligned} V &= \int_0^1 \delta(\psi)\,d\psi \\ &= \bar{\delta}, \end{aligned}$$

because the stream has unit depth. Thus,

$$\bar{\delta} = 2(\tfrac{4}{3}\epsilon)^{\frac{1}{2}}(1+\tfrac{3}{8}\epsilon-\tfrac{897}{3200}\epsilon^2)+O(\epsilon^{\frac{7}{2}}). \tag{25}$$

We can relate this quantity to the volume enclosed under the solitary wave by considering the conservation equation for volume in stationary co-ordinates (x^*, y^*)

$$\frac{\partial Q}{\partial x^*}+\frac{\partial \eta}{\partial t} = 0.$$

If Q and η are functions of x alone, where $x = x^*+Ut$, we can integrate the equation to give

$$\int_{-\infty}^{\infty} Q\,dt = \int_{-\infty}^{\infty} (\eta(x)-1)\,dx.$$

The quantity on the left-hand side is defined to be the drift volume, thus we have

$$\bar{\delta} = \int_{-\infty}^{\infty} (\eta(x)-1)\,dx. \tag{26}$$

That is, the drift volume (the mean drift) is equal to the volume under the solitary wave. If we substitute our expression for η, equation ($20a$), into (26), we find that we recover (25), thus providing a check on the result.

5. Ninth-order solitary wave solution

We have the governing exact operator equation (10), rewritten for the solitary wave

$$\eta^2 - \eta(F^2+2) + (2F^2+1) = F^2.\mathscr{D}(\eta), \tag{27}$$

where $\mathscr{D}(\eta)$ is the operator $[\sin \eta D]\, I[(\eta D/\sin \eta D)\,(1/\eta)]^2$. From the solutions of Boussinesq, Laitone and Grimshaw, it appears reasonable to express all solutions for the solitary wave profile in a series, each term of which is proportional to a product of some power of the amplitude and a power of $\operatorname{sech}^2 \alpha x$. That is,

$$\eta = 1 + \sum_{i=1}^{\infty} \sum_{j=1}^{i} a_{ij}\epsilon^i(\operatorname{sech}^2 \alpha x)^j, \tag{28}$$

where the a_{ij} constitute an array of unknown coefficients. We see that in (27) and (28) we have three parameters, two of which, F^2 and α, are expressible in an as yet unknown series in terms of the third, the amplitude ϵ.

If we replace the assumed form of solution (28) by a series in which α^2 is the expansion parameter, we see that all operations on η of addition, multiplication, inversion and differentiation of even order exhibit closure to give a series of similar form. Thus we write

$$\eta = 1 + \sum_{i=1}^{\infty} \sum_{j=1}^{i} b_{ij}\alpha^{2i} \operatorname{sech}^{2i} \alpha x, \tag{29}$$

where b_{ij} is an as yet undetermined coefficient array. We also simplify (27) by solving for η:

$$\eta = 1 + \tfrac{1}{2}F^2 - \tfrac{1}{2}F^2[1 + (4/F^2)\,(\mathscr{D}(\eta) - 1)]^{\frac{1}{2}}. \tag{30}$$

Now, we substitute Stokes's exact result (Lamb 1932, p. 425) for F^2

$$F^2 = \tan 2\alpha/2\alpha, \tag{31}$$

which can be expanded as a series in α^2. Thus we rewrite (30) as

$$\eta = 1 + \frac{1}{2}\frac{\tan 2\alpha}{2\alpha}\left\{1 - \left[1 + 4\frac{2\alpha}{\tan 2\alpha}(\mathscr{D}(\eta) - 1)\right]^{\frac{1}{2}}\right\}. \tag{32}$$

If we substitute (29) into this equation we can solve for the unknown coefficients b_{ij} by equating terms in each power of α^2 and each power of $\operatorname{sech}^2 \alpha x$. This was done on a computer as far as the ninth order, i.e. α^{18}. The limitation to this order was necessary to keep computation time within reasonable bounds.

With the coefficients b_{ij} calculated, substituting $x = 0$ gives a series expression for ϵ in terms of α^2, which can be subsequently reversed and substituted into (31). By substituting the results into (9) and (7), ninth-order expressions for the velocity at any point in the fluid are obtained, each containing 285 coefficients. These are then used to calculate expansions for the surface, mean and bottom drift. Also, by substituting into (29), the expansion for the wave profile is obtained. Because of the large amount of labour involved, these operations were performed by computer.

Quantity	Multiplicand of series	Term ϵ^0	ϵ^1	ϵ^2	ϵ^3	ϵ^4	ϵ^5	ϵ^6	ϵ^7	ϵ^8	ϵ^9
F^2	1·0	1·0000000	1·0000000	−0·0500000	−0·0428571	−0·0342857	−0·0315195	−0·0292784	−0·0268451	−0·0302634	−0·0219347
α	$(\frac{3}{4}\epsilon)^{\frac{1}{2}}$	1·0000000	−0·625000	0·554688	−0·561535	0·567095	−0·602969	0·624914	−0·670850	0·700371	—
η	1.0	1.0	—	—	—	—	—	—	—	—	—
	$\operatorname{sech}^2 \alpha x$	—	1.0	−0·75000	0·6250	−1·36817	1·86057	−2·57413	3·4572	−4·6849	6·191
	s^4	—	—	0·75000	−1·8875	3·88033	−7·45136	13·2856	−22·782	37·670	−60·57
	s^6	—	—	—	1·2625	−4·68304	12·7637	−31·1191	68·258	−139·28	269·84
	s^8	—	—	—	—	2·17088	−11·4199	40·1068	−116·974	301·442	−712·125
	s^{10}	—	—	—	—	—	4·24687	−28·4272	120·490	−411·416	1217·98
	s^{12}	—	—	—	—	—	—	8·7280	−71·057	355·069	−1384·37
	s^{14}	—	—	—	—	—	—	—	18·608	−180·212	1023·07
	s^{16}	—	—	—	—	—	—	—	—	41·412	−450·29
	s^{18}	—	—	—	—	—	—	—	—	—	90·279
$\delta\ (\psi=0)$	$(\frac{16}{3}\epsilon)^{\frac{1}{2}}$	1·000000	0·3750	−0·546979	−0·0748352	−0·0433862	−0·0226978	−0·0430576	−0·0170864	−0·362114	—
δ (mean)	$(\frac{16}{3}\epsilon)^{\frac{1}{2}}$	1·000000	0·3750	−0·280313	−0·1081693	−0·0630909	−0·0530746	−0·0420570	−0·0399407	−0·0350891	—
$\delta\ (\psi=1)$	$(\frac{16}{3}\epsilon)^{\frac{1}{2}}$	1·000000	0·3750	0·253022	0·0537308	0·0860583	0·0511546	0·0636512	0·0572738	0·0724525	—

TABLE 1. Coefficients of series expansions in terms of ϵ for wave speed, straining parameter, surface profile and drift.

The resulting series, for wave speed, wave profile and drift (bottom, surface and mean), are shown in table 1, which gives all coefficients correct to six figures for the expansions in terms of ϵ, the wave amplitude.

Shanks (1955) has shown how some divergent and slowly convergent series may be transformed so as to converge very rapidly to a limit. This is done by repeated use of the transform

$$e_{m+1}(S_n) = \frac{e_m(S_{n-1}) \cdot e_m(S_{n+1}) - e_m^2(S_n)}{e_m(S_{n-1}) + e_m(S_{n+1}) - 2e_m(S_n)}, \tag{33}$$

where $e_i(S_j)$ is the ith transform of the sum of the first j terms of a sequence, and where $e_0(S_j) = S_j$ for all j.

These transforms were applied numerically to the series shown in table 1. Results are shown and discussed in the next section.

6. Results

In table 1 we see that the coefficients in the expansion for F^2, the non-dimensional wave speed squared, are all negative after the first, and decrease in magnitude until the eighth-order term, at which a small oscillation in magnitude begins. We may notice that the coefficients agree with Long's fifth-order result. In figure 1 the results of this expansion are plotted for the different orders of approximation from one to nine, as well as the results obtained by applying a Shanks transform to the complete ninth-order solution. This differed from the transformed seventh-order solution by a minute amount, indicating that the transformed solutions have converged.

Each curve is plotted as far as the line $F^2 = 2\epsilon$, which, to satisfy the energy equation on the free surface, corresponds to the wave having zero velocity at the crest: the wave of greatest height. The point at which the transformed curve crosses this line has $\epsilon = 0{\cdot}85$, $F^2 = 1{\cdot}70$ ($F = 1{\cdot}30$); these then, are the values of amplitude and wave speed for the solitary wave of greatest height. This is comparable with an amplitude of 0·83 as obtained by McCowan (1894) and Lenau (1966), and the recent result of Strelkoff (1971), who obtained 0·85. Further support for the result of the present work is given by the fact that the curve of F^2 as a function of ϵ has a maximum at $\epsilon = 0{\cdot}85$; this is what we expect if the rounded crest is to 'snap through' to become the cusped crest of the highest wave, at which an increase in amplitude occurs for an infinitesimal increase in wave speed.

The expansion for the straining parameter α agrees with Grimshaw's third-order result. Beyond the third term it continues to be oscillatory; this may lead to rather inaccurate results for large amplitude waves, as can be seen in figure 2. For this type of divergence, the Shanks transform appears to give particularly powerful results; the transform of the complete nine terms is shown on figure 2 as well – the usefulness of the transform is immediately apparent.

If we are to examine the actual wave profiles, exact results are available for comparison, including those obtained numerically by Byatt-Smith (1970). Figure 3 shows the results of the present work compared with three waves computed by him. The ninth-order (untransformed) solution is perfectly adequate

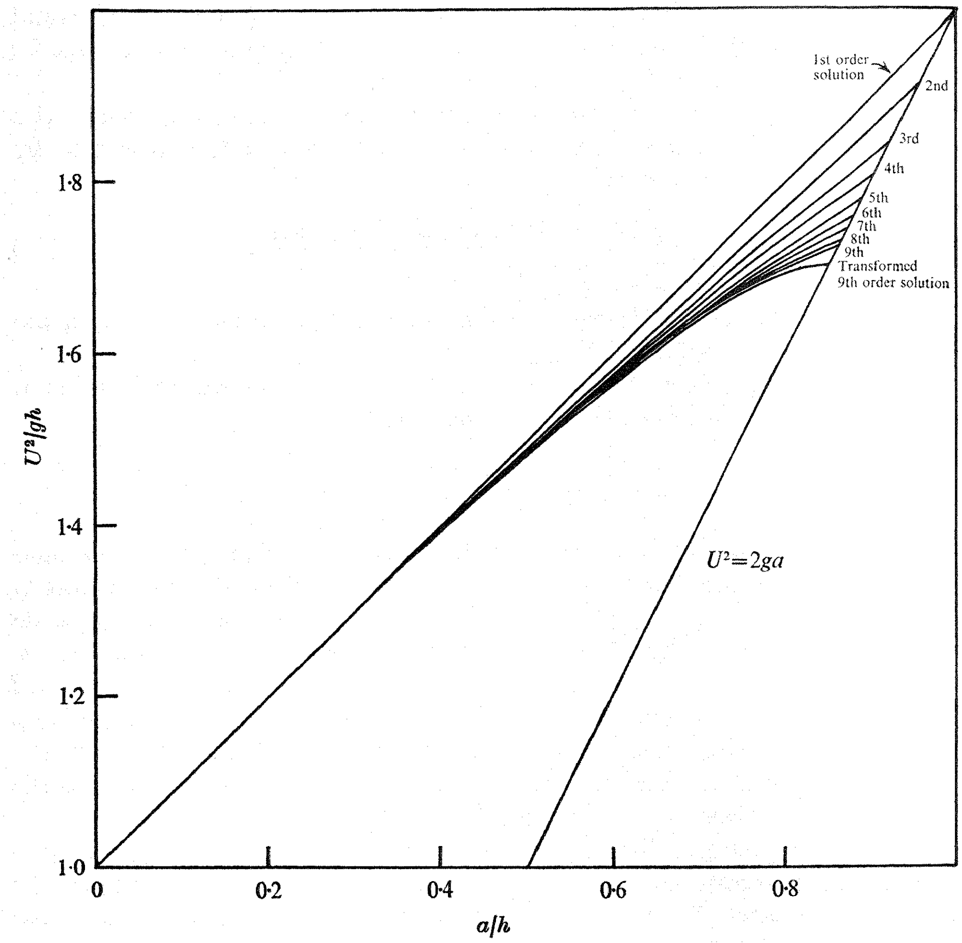

FIGURE 1. Wave speed. Solutions from first- to ninth-order and the transformed ninth-order solution are shown.

for wave amplitudes up to 0·5, but for higher waves the incorrect value of α, as shown by figure 2, gives a significantly greater rate of decay than the exact numerical results.

For the transformed ninth-order solution, as plotted on figure 3, the transformed value of the straining parameter α was computed, and for each value of x, the coefficients in the expression for the wave height at that point were calculated. Because the coefficients of each term (table 1) were often quite large, in some parts of the wave the sequence exhibited certain irregularities, thus poor results were obtained by a full $e_4(S_5)$ transformation. However, the seventh-, eighth- and ninth-order solutions were close together for all x and exhibited the same behaviour throughout, so a single transform was used on these three values, that is $e_1(S_8)$, to obtain the plotted curves. These agree very closely with the exact results over almost the whole range of amplitudes. For an amplitude of 0·752, however, some differences are apparent; these grow very quickly with amplitude

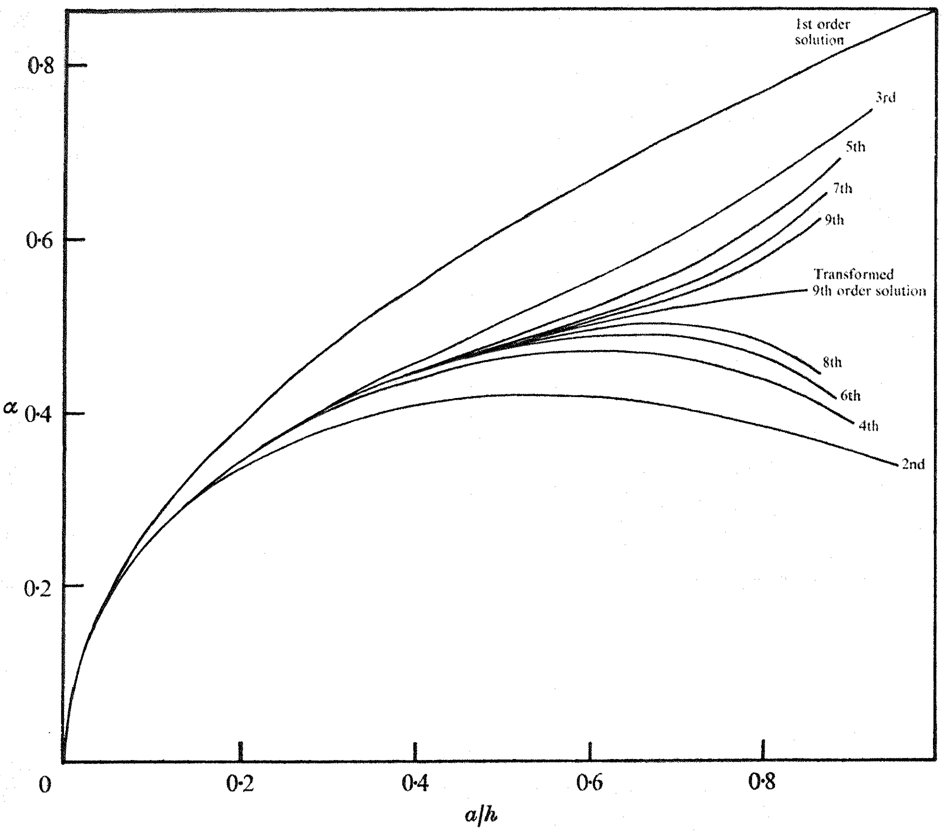

FIGURE 2. Straining parameter α. Solutions from first- to ninth-order and the transformed ninth-order solution are shown. Each curve is plotted up to the amplitude for which the velocity at the crest is zero.

at this height, until for the maximum amplitude of 0·85, the wave obtained bears no resemblance to the sharp-crested wave of greatest height.

Figure 4 shows the results for the expressions for fluid particle drift due to the passage of a solitary wave. From table 1 we see that the coefficients after the second-order term are all negative for the bottom and mean drift and all positive for the surface drift. Because of this we can see how Price's second-order drift result differed so markedly from the bottom drift computed by him using Lenau's results for the wave of greatest height. In fact for this wave, the velocity at the crest is zero, the fluid is carried along with the wave and the surface drift must be infinite. The upward-tending curve of the ninth-order expression indicates this. We may note that variation of drift with depth is obtained at third and higher orders; second-order theory predicts uniform drift throughout.

When these results were to be transformed, it was found that each (bottom, mean and surface drift) series exhibited certain irregularities, so as to invalidate the use of transforms as used above. Hence, no transformed results are presented; the curves shown on figure 4 are the ninth-order untransformed solutions. We can say, however, that surface drift, and hence the transport of contaminants,

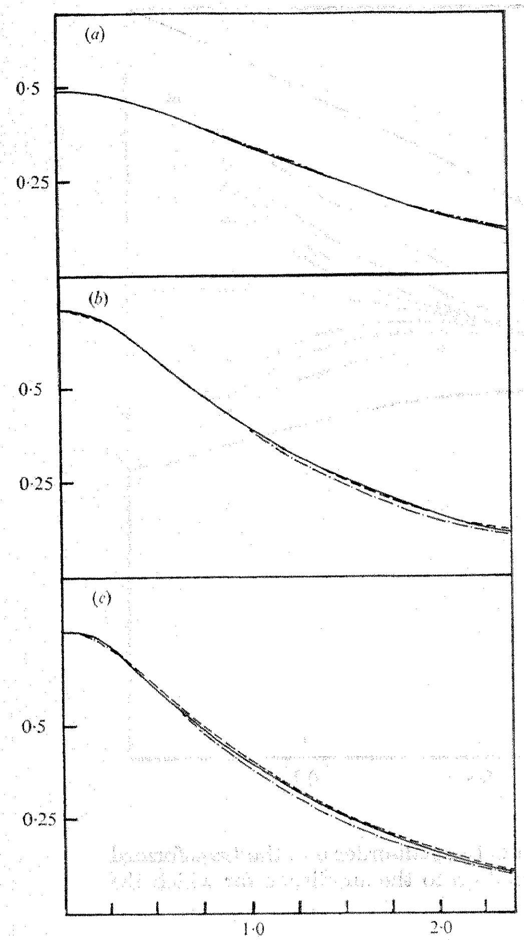

FIGURE 3. Comparison of wave profiles with those of Byatt-Smith. Vertical exaggeration is 2:1. —·—, ninth-order solution; ---, ninth-order solution transformed; ——, Byatt-Smith. (a) $a/h = 0{\cdot}492$. (b) $a/h = 0{\cdot}716$. (c) $a/h = 0{\cdot}752$.

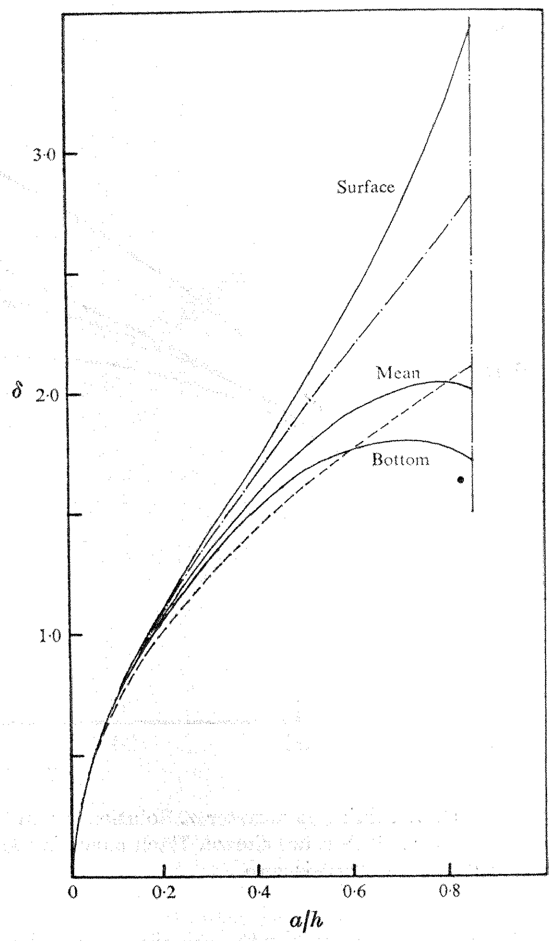

FIGURE 4. Drift: ---, first-order solution; —·—, second-order solution; ——, ninth-order solution (surface, mean, and bottom drift as shown). All curves are plotted as far as the line $a/h = 0{\cdot}85$, corresponding to the solitary wave of greatest height. ●, Price's result for this wave (after Lenau).

has been shown to be much larger than previously calculated, and in view of the relative amount of bottom, mean and surface drift, that this enhanced drift would be confined to a region relatively close to the surface.

7. Concluding remarks

An exact operator equation has been solved to give numerical coefficients for an assumed form of solution to the solitary wave, which were calculated to the ninth order. This solution was used to obtain an estimate of the solitary wave of greatest height and refined estimates of drift due to solitary waves. The standard of comparison for the wave profiles was the work of Byatt-Smith, the numerical solution of an exact integro-differential equation. Results from the present work were seen to agree closely, even up to waves of amplitude 0·75; for the wave of greatest height, of amplitude 0·85, however, the present method was of no use in calculating the profile.

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