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Wave forces on vertical bodies of revolution

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The axisymmetry of a body which is diffracting water waves may be exploited to give a line integral equation to be solved for the scattered wave field and forces on the body. Each term in a previously established surface integral equation is shown to be expressible as a Fourier series, which is then integrated once analytically. The resulting one-dimensional equation is shown to possess singularities, previously ignored by Black (1975). This equation, with series transformations and subtraction of singularities such that all series are quickly convergent and that it has to be solved only along a curve, reduces computational effort by some three orders of magnitude. Results obtained by this method give good agreement with previous analytical and experimental results, even if a rather coarse numerical approximation is used.

1. Introduction

In 1950, John obtained a Green's function for a fluid layer bounded below by a horizontal surface and above by a free surface on which (linear) waves were propagating. This function has been used by several workers, notably Garrison & Chow (1972), to set up integral equations for the unknown magnitude of an assumed source distribution on the surface of a body immersed in the fluid. After numerical solution of this equation the scattered potentials, velocities and pressures may be obtained.

Black (1975) studied the scattering due to bodies which are axisymmetric about a vertical axis and reduced the surface integral equation of Garrison & Chow to a one-dimensional equation. In §2 of his paper Black maintained that his Green's function was non-singular. Examination of his equation (2.4) shows that this is not the case, and that it possesses a logarithmic singularity. It is surprising that this did not appear in his subsequent numerical calculations, but it is hinted at in his §5, where it is noted that the convergence rate of the series term in his Green's function was 'roughly $1/n$'. In §2 below we derive his function, showing how it is obtained simply from John's, and show that the terms at the singularity go as $r^n \cos(n\theta)/n$, as $r \to 1$, $\theta \to 0$. The sum of these terms to infinity gives $\ln[(1-r)^2+\theta^2]$, showing the nature of the singularity.

In §3 we take John's equations, as used by Garrison & Chow, and express them as Fourier series in terms of the azimuthal angle about the vertical axis of symmetry, obtaining a one-dimensional integral equation analogous to that of Black. The singularity in the kernel function is subtracted in §4, using a transformation of the series by

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subtraction and separate summation of asymptotic terms. After a second subtraction of the six singularities, these are integrated analytically, so that all subsequent numerical approximation is with smooth bounded functions. Subsequently the integral equation is approximated by a matrix equation which may be solved numerically. In §5 the results of this, with transformation of series and subtraction of singularities similar to those in §4, are used to set up matrix expressions for forces, moments and pressures on the body. Sample results for a truncated circular cylinder are given in §6.

Here we give the equations for a body of arbitrary shape, as used by Garrison & Chow: consider a fixed body of arbitrary shape and position immersed in a fluid of depth h above a horizontal surface. We introduce a rectangular co-ordinate system with the origin on the sea bed, x in the direction of propagation of waves, y perpendicular to this in the plane of the sea bed, and z vertically upwards. The wave train has amplitude a (wave height $= 2a$), wavelength λ, wavenumber $k = 2\pi/\lambda$, and wave frequency $\sigma = 2\pi/T$, where T is the period. Using irrotational flow theory and the linearized wave approximation, we have the following equations for a body of arbitrary shape.

A velocity potential Φ exists:

$$\Phi = \mathrm{Re}\{\phi(x, y, z)\, e^{-i\sigma t}\}, \tag{1.1}$$

where t is time and ϕ is a complex variable which may be split into incident and scattered parts

$$\phi = \phi_i + \phi_s, \tag{1.2}$$

where the incident term is well known from linear wave theory,

$$\phi_i = \frac{-ga}{\sigma}\frac{\cosh kz}{\cosh kh} e^{ikx}, \tag{1.3}$$

in which g is the gravitational acceleration.

The scattered potential, as yet unknown, may be assumed to be generated by a distribution of sources with strength $f(X, Y, Z)$ over the immersed surface of the body:

$$\phi_s(x, y, z) = \frac{1}{4\pi}\int_A f(X, Y, Z)\, G(x, y, z | X, Y, Z)\, dA, \tag{1.4}$$

where the variables of integration (X, Y, Z) are the co-ordinates of points on the wetted surface of the body, denoted by A. G is a Green's function, determined by John (1950):

$$G = C_0 \cosh kz \cosh kZ[Y_0(kq) - iJ_0(kq)] + 4\sum_{m=1}^{\infty} C_m \cos(\mu_m z)\cos(\mu_m Z)\, K_0(\mu_m q),$$

where

$$C_0 = \frac{2\pi(\nu^2 - k^2)}{(k^2 - \nu^2)\,h + \nu}, \quad C_m = \frac{\mu_m^2 + \nu^2}{(\mu_m^2 + \nu^2)\,h - \nu},$$

$$\nu = \sigma^2/g = k \tanh kh,$$

the μ_m are positive real roots of $\nu + \mu_m \tan \mu_m h = 0$, and

$$q^2 = (x - X)^2 + (y - Y)^2. \tag{1.5}$$

An alternative integral expression for G has been determined (see John 1950) but it is not as convenient for the present work because the independent variables are not as simply separated.

The unknown source strength distribution must be such as to satisfy the boundary condition on the body surface, that no fluid passes through the surface. Combining (1.2)–(1.4) and differentiating with respect to n, the local outwardly directed normal to the surface, to obtain the normal fluid velocity, we have

$$-2\pi f(x,y,z)+\frac{\partial}{\partial n}\int_A f(X,Y,Z)\,G(x,y,z|X,Y,Z)\,dA(X,Y,Z)+4\pi\frac{\partial\phi_i}{\partial n}(x,y,z)=0. \tag{1.6}$$

Equation (1.6) is to be satisfied at all points on the wetted body surface $A(x,y,z)$. Once this integral equation has been solved for f, this is substituted into (1.4) to give $\phi_s(x,y,z)$. From this, other physical quantities are easily calculated:

$$p/\rho = \partial\Phi/\partial t = \mathrm{Re}\{-i\sigma(\phi_i+\phi_s)\,e^{-i\sigma t}\}, \tag{1.7}$$

where p is the pressure at any point and ρ is fluid density;

$$\eta = \mathrm{Re}\{(-i\sigma/g)\,(\phi_i+\phi_s)\,e^{-i\sigma t}\} \quad \text{on} \quad z=h, \tag{1.8}$$

where η is the free-surface elevation relative to the undisturbed level. Fluid velocities $\mathbf{u}$ are given by

$$\mathbf{u} = -\nabla\Phi = \mathrm{Re}\{-\nabla(\phi_i+\phi_s)\,e^{-i\sigma t}\}. \tag{1.9}$$

The total force exerted on the body $\mathbf{F}$ and the moment of this force $\mathbf{M}$ are given by

$$\mathbf{F}(t) = -\int_A p\hat{\mathbf{n}}\,dA, \tag{1.10}$$

$$\mathbf{M}(t) = -\int_A p(\mathbf{r}\times\hat{\mathbf{n}})\,dA, \tag{1.11}$$

where $\hat{\mathbf{n}}$ is a unit outward normal vector on A and $\mathbf{r}$ is the vector from the point about which moments are taken.

2. The Green's function as a Fourier series

In this section we examine the function G defined in (1.5), convert to cylindrical co-ordinates and write it as a trigonometric series. Each term in the series is shown to have a logarithmic singularity.

Introducing cylindrical co-ordinates we write

$$r^2 = x^2+y^2, \quad R^2 = X^2+Y^2, \quad \tan\theta = y/x, \quad \tan\Theta = Y/X;$$

then $q^2 = R^2+r^2-2Rr\cos(\theta-\Theta)$. Making use of Graf's addition theorem (Watson 1944, § 11.3) gives

$$\left.\begin{aligned} J_0(kq) &= \sum_{j=-\infty}^{\infty} J_j(kR)\,J_j(kr)\cos j(\theta-\Theta),\\ Y_0(kq) &= \sum_{j=-\infty}^{\infty} Y_j\left(k\frac{r}{R}\right)J_j\left(k\frac{R}{r}\right)\cos j(\theta-\Theta),\\ K_0(\mu_m q) &= \sum_{j=-\infty}^{\infty} K_j\left(\mu_m\frac{r}{R}\right)I_j\left(\mu_m\frac{R}{r}\right)\cos j(\theta-\Theta), \end{aligned}\right\} \tag{2.1}$$

where the upper of the alternative arguments is used if $r \geqslant R$ and the lower otherwise. In each of the three series the $-j$th term is equal to the jth term. Introducing a Kronecker delta we may write

$$J_0(kq) = \sum_{j=0}^{\infty} (2-\delta_{j0})\, J_j(kR)\, J_j(kr) \cos j\,(\theta - \Theta), \tag{2.2}$$

the other two expressions being transformed in the same way. Substituting into (1.5) we have a doubly infinite series, but one in which all independent variables have been separated:

$$G = C_0 \cosh kz \cosh kZ \sum_{j=0}^{\infty} (2-\delta_{j0})\, J_j\!\left(k\, {}^{R}_{r}\right) \left(Y_j\!\left(k\, {}^{r}_{R}\right) - iJ_j\!\left(k\, {}^{r}_{R}\right)\right) \cos j\,(\theta - \Theta)$$
$$+ 4 \sum_{m=1}^{\infty} C_m \cos \mu_m z \cos \mu_m Z \sum_{j=0}^{\infty} (2-\delta_{j0})\, K_j\!\left(\mu_m\, {}^{r}_{R}\right) I_j\!\left(\mu_m\, {}^{R}_{r}\right) \cos j\,(\theta - \Theta). \tag{2.3}$$

This is similar to Black's equation (2.4), the only difference being in the angular dependence: (2.3) contains $\cos j(\theta - \Theta)$, which is $\cos j\theta \cos j\Theta + \sin j\theta \sin j\Theta$, of which Black's expression contains the first term.

The nature of the singularity can be established by considering (2.3) as

$$(r, \theta, z) \to (R, \Theta, Z).$$

Examining the first part, all terms of the series in j are finite. However for the doubly infinite part this is not so. Let

$$G_{j0} = C_0 \cosh(kz) \cosh(kZ)\, J_j\!\left(k\, {}^{R}_{r}\right) \left(Y_j\!\left(k\, {}^{r}_{R}\right) - iJ_j\!\left(k\, {}^{r}_{R}\right)\right) \tag{2.4a}$$

and

$$G_{jm} = 4\, C_m \cos(\mu_m z) \cos(\mu_m Z)\, K_j\!\left(\mu_m\, {}^{r}_{R}\right) I_j\!\left(\mu_m\, {}^{R}_{r}\right). \tag{2.4b}$$

Then we can write (2.3) as

$$G = \sum_{j=0}^{\infty} \left(G_{j0} + \sum_{m=1}^{\infty} G_{jm}\right) (2-\delta_{j0}) \cos j\,(\theta - \Theta). \tag{2.4c}$$

As $m \to \infty$, it is easily shown from (1.5) that $\mu_m \to m\pi/h + O(m^{-1})$, $C_m \to 1/h + O(m^{-2})$, while the product of the Bessel functions goes as

$$\frac{h}{2\pi}(Rr)^{-\frac{1}{2}} \frac{1}{m} \exp(-m\pi|R-r|/h).$$

Thus $\quad G_{jm} \to 2(m\pi)^{-1}(Rr)^{-\frac{1}{2}} \cos m\pi z/h \cos m\pi Z/h \exp(-m\pi|R-r|/h)$

$$\to (m\pi)^{-1}(Rr)^{-\frac{1}{2}} \exp(-m\pi|R-r|/h)\,(\cos m\pi(Z+z)/h + \cos m\pi(Z-z)/h).$$

Except for the cases $z \to Z \to 0$ or h, which will be treated in the next section, it is the second term which is important as $z \to Z$, giving

$$G_{jm} \to (m\pi)^{-1} \exp(-m\pi|R-r|/h) \cos m\pi(Z-z)/h$$

and

$$\sum_{m=1}^{\infty} G_{jm} \sim \sum_{m=1}^{\infty} \frac{1}{m} \exp(-m\pi|R-r|/h) \cos m\pi(Z-z)/h$$
$$\sim -\tfrac{1}{2} \ln\left[1 - 2\exp(-\pi|R-r|/h) \cos \pi(Z-z)/h + \exp(-2\pi|R-r|/h)\right]$$

(Jolley 1961, §536), which as $z \to Z$ and $r \to R$ gives

$$\sum_{m=1}^{\infty} G_{jm} \sim \tfrac{1}{2}\ln[(R-r)^2+(Z-z)^2], \tag{2.5}$$

showing the logarithmic nature of the singularity.

3. Reduction of integral equation

In this and succeeding sections we limit our attention to bodies of revolution, as shown in figure 1, formed by rotating a generating curve about the vertical axis. We use the integral equation (1.6) together with the Fourier expansion of the Green's function (2.4) to obtain a Fourier series, the coefficients of which are integral equations valid on the arc AA' of figure 1. Subsequently it is shown that only the zeroth and first terms need be solved to give the forces on the body.

The source strength f is a function of position on the body and may be written as $f(s,\theta)$, where the co-ordinate $s(r,z)$ specifies a point on the curve AA'. As the problem is symmetrical about the x axis, we may expand f in a cosine series:

$$f(s,\theta) = \sum_{l=0}^{\infty} f_l(s)\cos l\theta. \tag{3.1}$$

Substituting (3.1) and (2.4) into (1.6) we have

$$-2\pi\sum_{l=0}^{\infty} f_l(s)\cos l\theta + \frac{\partial}{\partial n}\int_A \left[\sum_{l=0}^{\infty} f_l(S)\cos l\Theta\right] \times\left[\sum_{j=0}^{\infty}(2-\delta_{j0})\left(G_{j0}+\sum_{m=1}^{\infty}G_{jm}\right)\cos j(\theta-\Theta)\right]dA + 4\pi\frac{\partial\phi_i}{\partial n} = 0. \tag{3.2}$$

We also have (1.3) for ϕ_i, rewritten in cylindrical co-ordinates:

$$\phi_i = -\frac{ga}{\sigma}\frac{\cosh kz}{\cosh kh}e^{ikr\cos\theta}.$$

It is easily shown that the operator $\partial/\partial n$ for an axisymmetric body traced anticlockwise is

$$\partial/\partial n \equiv z'\partial/\partial r - r'\partial/\partial z,$$

where $z' = dz/ds$ and $r' = dr/ds$. Performing the operation we have

$$\begin{aligned}\frac{\partial\phi_i}{\partial n} &= \frac{gak\,e^{ikr\cos\theta}}{\sigma\cosh kh}(r'\sinh kz - iz'\cos\theta\cosh kz)\\ &= \frac{gak}{\sigma\cosh kh}(r'\sinh kz - iz'\cos\theta\cosh kz)\sum_{l=0}^{\infty}(2-\delta_{l0})\,i^l J_l(kr)\cos l\theta\\ &= \frac{gak}{\sigma\cosh kh}\sum_{l=0}^{\infty}(2-\delta_{l0})\,i^l[r'\sinh(kz)J_l(kr) - z'\cosh(kz)J_l'(kr)]\cos l\theta,\end{aligned} \tag{3.3}$$

which shows that the incident velocity field has been expressed as a Fourier series with coefficients that are functions of r and z.

Now we can rewrite the second term in (3.2), with $dA = R\,d\Theta\,dS$, where dS is an element of AA', so that it becomes

$$\frac{\partial}{\partial n}\int_S \sum_{l=0}^{\infty} f_l(S)\sum_{j=0}^{\infty}\left[\left(G_{j0}+\sum_{m=1}^{\infty}G_{jm}\right)(2-\delta_{j0})\,R\int_{-\pi}^{\pi}\cos l\Theta\cos j(\theta-\Theta)\right]d\Theta\,dS. \tag{3.4}$$

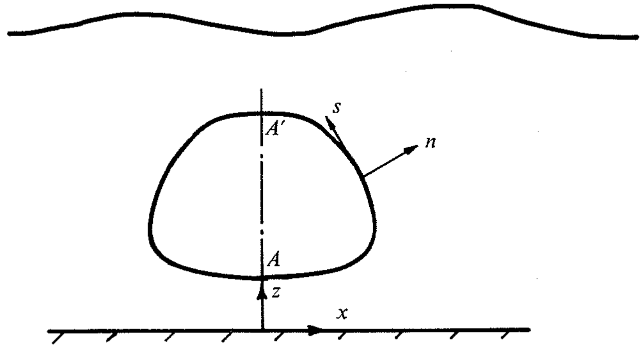


FIGURE 1. Section through axisymmetric body with co-ordinate systems. Equation (3.6) is to be satisfied at all points on the generator AA'.

Integrating with respect to Θ we have

$$2\pi \sum_{l=0}^{\infty} \left[\frac{\partial}{\partial n}\int_S f_l(S)\, R\left(G_{l0}+\sum_{m=1}^{\infty} G_{lm}\right) dS\right] \cos l\theta. \tag{3.5}$$

From (3.1)–(3.3) and (3.5) we see that the integral equation can be expressed as a Fourier series, with each term satisfying the equation. Thus for each l we can write

$$-f_l(s)+\int_S f_l(S)\, R\left(z'\frac{\partial}{\partial r}-r'\frac{\partial}{\partial z}\right)\left(G_{l0}+\sum_{m=1}^{\infty} G_{lm}\right) dS$$
$$+\frac{2(2-\delta_{l0})\, gaki^l}{\sigma \cosh kh}\,[r' \sinh (kz)\, J_l(kr)-z' \cosh (kz)\, J_l'(kr)] = 0, \quad l = 0,1,2,\ldots, \tag{3.6}$$

and we now have a series of one-dimensional integral equations in place of the original surface integral equation.

4. Solution

4.1. *Numerical approximation by a matrix equation*

Equation (3.6), valid on the curve which generates the body, is solved numerically by dividing the curve into N line segments and solving the equation at the centre of each. We define ψ_i to be the complex dimensionless source strength at the centre of element i,

$$\psi_i = f_l(s_i)\frac{\sigma(1+\delta_{l0})\cosh kh}{4gak}\, e^{\frac{1}{2}l\pi i}, \tag{4.1}$$

and b_i to be the dimensionless quantity

$$b_i = z_i' \cosh (kz_i)\, J_l(kr_i)-r_i' \sinh (kz_i)\, J_l(kr_i) \tag{4.2}$$

and assume that ψ is linear over an element. Then (3.6) can be written as

$$-\psi_i+\sum_{j=1}^{N} \psi_j a_{ij} = b_i, \quad i = 1,\ldots,N, \tag{4.3}$$

where the element a_{ij} is the integrated contribution of the kernel function over element j:

$$a_{ij} = \int_{S_j} R\left(z_i'\frac{\partial}{\partial r} - r_i'\frac{\partial}{\partial z}\right)\left(G_{l0} + \sum_{m=1}^{\infty} G_{lm}\right) dS. \tag{4.4}$$

Obviously (4.3) can be written as a matrix equation

$$[a_{ij} - \delta_{ij}][\psi_i] = [b_i], \tag{4.5}$$

which must be solved for $[\psi_i]$.

4.2. *Generation of matrix elements*

Now we need to obtain an expression for a_{ij} from (2.4) and (4.4). The first part of the kernel function is simply handled, for G_{l0} is continuous and finite. The real part of $\partial G_{l0}/\partial r$ has a discontinuity at $r = R$, however any numerical errors associated with this will be swamped by numerical treatment of the singularity, so that we use a midpoint approximation throughout and can write

$$\int_{S_j} R\frac{\partial G_{l0}}{\partial n} dS = kC_0 R_j L_j \cosh kZ_j \left[z_i' \cosh kz_i \begin{Bmatrix} J_l(kR_j)\,(Y_l'(kr_i) - iJ_l'(kr_i)) \\ J_l'(kr_i)\,(Y_l(kr_i) - iJ_l(kr_i)) \end{Bmatrix}\right.$$
$$\left. - r_i' \sinh kz_i \begin{Bmatrix} J_l(kR_j)\,(Y_l(kr_i) - iJ_l(kr_i)) \\ J_l(kr_i)\,(Y_l(kR_j) - iJ_l(kR_j)) \end{Bmatrix}\right], \tag{4.6}$$

where integration has been approximated over the element j of length L_j. In this expression and subsequently, the upper alternative in the curly brackets is to be used if $R_j < r_i$, the lower if $R_j > r_i$ and the mean of the two if $R_j = r_i$.

The next term, $\int \Sigma\,(\partial G_{lm}/\partial n)\,R\,dS$, is much more difficult to handle, as the series does not converge at some points. Performing the differentiation we have

$$\frac{\partial}{\partial n}\Sigma G_{lm} = \sum_{m=1}^{\infty} 4\mu_m C_m \cos\mu_m Z \begin{Bmatrix} I_l(\mu_m R) \\ K_l(\mu_m R) \end{Bmatrix} \left(z' \cos\mu_m z \begin{Bmatrix} K_l'(\mu_m r) \\ I_l'(\mu_m r) \end{Bmatrix} + r' \sin\mu_m z \begin{Bmatrix} K_l(\mu_m r) \\ I_l(\mu_m r) \end{Bmatrix}\right), \tag{4.7}$$

which we have shown in §2 to be non-convergent at $(r, z) = (R, Z)$. If the asymptotic form of G_{lm} as $m \to \infty$ is denoted by G^*_{lm}, after some manipulation we can show that

$$\frac{\partial G^*_{lm}}{\partial n} = -2h^{-1}(Rr)^{-\frac{1}{2}} \exp(-m\pi|R-r|/h)\left[\left(-\frac{\nu Zr'}{m\pi}\right)\sin\frac{m\pi Z}{h}\sin\frac{m\pi z}{h}\right.$$
$$+\left(\begin{Bmatrix}+\\-\end{Bmatrix}\frac{\nu zz'}{m\pi} - r' - \frac{\nu r'|R-r|}{m\pi} + \frac{4n^2-1}{8m\pi}\frac{h|R-r|}{Rr}r'\right)\cos\frac{m\pi Z}{h}\sin\frac{m\pi z}{h}$$
$$+\left(\begin{Bmatrix}+\\-\end{Bmatrix}\frac{\nu Zz'}{m\pi}\right)\sin\frac{m\pi Z}{h}\cos\frac{m\pi z}{h}$$
$$\left.+\left(\begin{Bmatrix}+\\-\end{Bmatrix}z' + \frac{\nu(r-R)z'}{m\pi} - z'\frac{h}{R}\frac{4l^2-1}{8m\pi} + \frac{z'h}{r}\frac{4l^2+3}{8m\pi} + \frac{\nu zr'}{m\pi}\right)\cos\frac{m\pi Z}{h}\cos\frac{m\pi z}{h}\right]$$
$$+ O(m^{-2}\exp(-m\pi|R-r|/h)). \tag{4.8}$$

By subtracting terms (4.8) from each term in (4.7) we have a series that converges everywhere as m^{-2} at least. Thus we can transform (4.7) to read

$$\Sigma\frac{\partial G_{lm}}{\partial n} = \Sigma\left(\frac{\partial G_{lm}}{\partial n} - \frac{\partial G^*_{lm}}{\partial n}\right) + \Sigma\frac{\partial G^*_{lm}}{\partial n}, \tag{4.9}$$

where the first sum can be computed with guaranteed convergence. The second sum, with terms as given in (4.8), has closed-form expressions, as given by Jolley (1961, §§499, 500, 536 and 540). Thus we have

$$\Sigma\frac{\partial G^*_{lm}}{\partial n} = h^{-1}(Rr)^{-\frac{1}{2}}\Bigg[\begin{Bmatrix}-\\+\end{Bmatrix}\frac{z'\gamma(\cos\Sigma-\gamma)}{1-2\gamma\cos\Sigma+\gamma^2}\begin{Bmatrix}-\\+\end{Bmatrix}\frac{z'\gamma(\cos\Delta-\gamma)}{1-2\gamma\cos\Delta+\gamma^2}$$

$$+\frac{r'\gamma\sin\Sigma}{1-2\gamma\cos\Sigma+\gamma^2}-\frac{r'\gamma\sin\Delta}{1-2\gamma\cos\Delta+\gamma^2}$$

$$+\left(\frac{\nu z'}{2\pi}(r-R)-\frac{hz'}{R}\frac{4l^2-1}{16\pi}+\frac{hz'}{r}\frac{4l^2+3}{16\pi}+\frac{\nu r'}{2\pi}(Z+z)\right)\ln(1-2\gamma\cos\Sigma+\gamma^2)$$

$$+\left(\frac{\nu z'}{2\pi}(r-R)-\frac{hz'}{R}\frac{4l^2-1}{16\pi}+\frac{hz'}{r}\frac{4l^2+3}{16\pi}+\frac{\nu r'}{2\pi}(z-Z)\right)\ln(1-2\gamma\cos\Delta+\gamma^2)$$

$$+\left(\begin{Bmatrix}-\\+\end{Bmatrix}\frac{\nu z'}{\pi}(Z+z)+\frac{\nu r'}{\pi}|R-r|\begin{Bmatrix}+\\-\end{Bmatrix}hr'\left(\frac{1}{r}-\frac{1}{R}\right)\frac{4l^2-1}{8\pi}\right)\tan^{-1}\left(\frac{\gamma\sin\Sigma}{1-\gamma\cos\Sigma}\right)$$

$$+\left(\begin{Bmatrix}-\\+\end{Bmatrix}\frac{\nu z'}{\pi}(Z-z)-\frac{\nu r'}{\pi}|R-r|\begin{Bmatrix}-\\+\end{Bmatrix}hr'\left(\frac{1}{r}-\frac{1}{R}\right)\frac{4l^2-1}{8\pi}\right)\tan^{-1}\left(\frac{\gamma\sin\Delta}{1-\gamma\cos\Delta}\right)\Bigg], \quad (4.10)$$

where $\gamma = \exp(-\pi|R-r|/h),\quad \Delta = \pi(Z-z)/h,\quad \Sigma = \pi(Z+z)/h,$

and we can show that each term in (4.10), except the $\tan^{-1}$ quantities, shows singular behaviour, either first order or logarithmic. Let S_6 represent the sum of the six singularities:

$$S_6 = \pi^{-1}(Rr)^{-\frac{1}{2}}\Bigg[\frac{z'(R-r)+r'(Z+z-2h)}{(R-r)^2+(Z+z-2h)^2}+\frac{z'(R-r)+r'(Z+z)}{(R-r)^2+(Z+z)^2}$$

$$+\frac{z'(R-r)-r'(Z-z)}{(R-r)^2+(Z-z)^2}+\frac{z'}{4r}\ln\frac{\pi^2}{h^2}((R-r)^2+(Z+z)^2)$$

$$+\left(\frac{z'}{4r}+\nu r'\right)\ln\frac{\pi^2}{h^2}((R-r)^2+(Z+z-2h)^2)$$

$$+\frac{z'}{4r}\ln\frac{\pi^2}{h^2}((R-r)^2+(Z-z)^2)\Bigg] \quad (4.11)$$

and
$$\Sigma\frac{\partial G^*_{lm}}{\partial n} = \left(\Sigma\frac{\partial G^*_{lm}}{\partial n}-S_6\right)+S_6,$$

where the term inside the brackets is everywhere finite and cntinuous, with the singularities subtracted. In our numerical approximation of the integral equation we have taken the source strength outside the integral (§4.1); to the same accuracy we can do the same with R, so that we may write

$$\int_{S_j} R\,\Sigma\frac{\partial G_{lm}}{\partial n}dS = R_j\int_{S_j}\Sigma\frac{\partial G_{lm}}{\partial n}dS$$

$$= R_j L_j\left[\Sigma\left(\frac{\partial G_{lm}}{\partial n}-\frac{\partial G^*_{lm}}{\partial n}\right)+\left(\Sigma\frac{\partial G^*_{lm}}{\partial n}-S_6\right)\right]+R_j\int_{S_j}S_6\,dS. \quad (4.12)$$

We can integrate the last term analytically, obtaining a finite contribution to a_{ij}.

Combining all the contributions we have a very long expression for the matrix coefficients a_{ij}, but one in which all series converge quickly and all terms are finite. This expression is given in the appendix.

4.3. *Solution of matrix equation*

Equation (4.5), which we are to solve for ψ_i, has a complex matrix $[a_{ij} - \delta_{ij}]$ multiplying a complex column vector $[\psi_i]$, the result being equal to a real column vector $[b_i]$. There are techniques available for separating the real and imaginary parts of the matrix and solving a series of matrix equations (see, for example, Hogben & Standing 1974). This seems unnecessarily complicated, for in the present work the matrices generated have heavily dominant leading-diagonal terms because of the δ_{ii} contributions, hence iterative techniques may be used. In the process of testing the present work, a Gauss–Seidel method with complex arithmetic was used. That is, if we have an approximation ψ_i^k to the solution after k iterations, a better approximation is had by substituting this into the re-arranged matrix equation (4.3):

$$\psi_i^{k+1} = \frac{b_i - \left(\sum\limits_{j=1}^{i-1} + \sum\limits_{j=i+1}^{N}\right)(\psi_j^k a_{ij})}{a_{ii} - 1}, \quad i = 1, 2, \ldots, N. \tag{4.13}$$

This process is repeated until it has converged.

4.4. *Estimate of saving with one-dimensional equation*

Consider a fixed sphere which is scattering waves. If the sphere has a radius R and the surface is divided into a number of planar elements of side length S, then the number of facets is approximately $4\pi R^2/S^2$. If the above one-dimensional method is used, the number of line segments is approximately $\pi R/S$. The amount of computational effort required to set up and solve the matrix is proportional to at least the square of the number of unknowns, hence the ratio of effort involved using the present method to that with a surface integral equation is $\simeq \frac{1}{16}(S/R)^2$. If we need the total force on a body we need to do the $l = 0$ and $l = 1$ cases, so the ratio is $\frac{1}{8}(S/R)^2$. If we assume $S/R \simeq \frac{1}{10}$, the ratio $\simeq \frac{1}{800}$.

5. Forces on body

Having solved for ψ_i and hence $f_n(s_i)$ we may now obtain the scattered potential ϕ_s by taking (1.4), with all terms expanded in Fourier series, and subsequently substituting into (1.7), (1.10) and (1.11). We have

$$\phi_s = \frac{1}{4\pi}\int_A fG\,dA$$

[equation (1.4)] and we have written

$$f(s, \Theta) = \sum_{l=0}^{\infty} f_l(s)\cos l\Theta, \quad dA = R\,d\Theta\,dS$$

[equation (3.1)] and $$G = \sum_{j=0}^{\infty} (2 - \delta_{j0})(G_{j0} + \Sigma G_{jm})\cos j(\theta - \Theta)$$

[equation (2.4)]. Substituting into (1.4) and integrating with respect to Θ, we obtain

$$\phi_s(r,\theta,z) = \sum_{l=0}^{\infty} \phi_{sl}(r,z)\cos l\theta,$$

where

$$\phi_{sl}(r,z) = \tfrac{1}{2}\int_S f_l(S)\,R\left[C_0\cosh(kz)\cosh(kZ)\left(Y_l\left(k\frac{R}{r}\right)J_l\left(k\frac{r}{R}\right) - iJ_l(kR)\,J_l(kr)\right)\right.$$
$$\left. +4\sum_{m=1}^{\infty} C_m\cos(\mu_m z)\cos(\mu_m Z)\,K_l\left(\mu_m\frac{r}{R}\right)I_l\left(\mu_m\frac{R}{r}\right)\right]dS. \quad (5.1)$$

Now we write this for an element i on the body, using the same degree of approximation as in §4:

$$\phi_{sl}(s_i) = \tfrac{1}{2}\sum_{j=1}^{N} f_l(s_i)\,c_{ij},$$

where

$$c_{ij} = R_j\int_{S_j}\left(G_{l0} + \sum_{m=1}^{\infty} G_{lm}\right)dS. \quad (5.2)$$

The calculation of c_{ij} presents the same problems as that of a_{ij}, for the integrand has a singularity because of the non-convergence of the series in m. In this case, however, it is logarithmic rather than first order, and the manipulations are somewhat shorter. Performing asymptotics and subtraction of singularities similar to those in §4, we obtain the expression for c_{ij} given in the appendix.

From (5.2) and (4.1) we have

$$\phi_{sl}(s_i) = \frac{2gaki^l}{\sigma(1+\delta_{l0})\cosh kh}\sum_{j=1}^{N}\psi_j c_{ij}, \quad (5.3)$$

and from §3 we have

$$\phi_i = -\frac{ga\cosh kz}{\sigma\cosh kh}e^{ikr\cos\theta},$$

which may be written as

$$\phi_i = \sum_{l=0}^{\infty}\phi_{il}\cos l\theta,$$

with

$$\phi_{il} = -\frac{2ga\cosh kz}{\sigma\cosh kh}\frac{i^l}{(1+\delta_{l0})}J_l(kr), \quad (5.4)$$

giving

$$\phi_l(s_i) = \phi_{il}(s_i) + \phi_{sl}(s_i)$$
$$= \frac{2gai^l}{\sigma(1+\delta_{l0})\cosh kh}\left(k\sum_{j=1}^{n}\psi_j c_{ij} - \cosh(kz_i)\,J_l(kr_i)\right) \quad (5.5)$$

for the lth component of the combined incident and scattered velocity potential.

From (1.7) and (1.10) we have

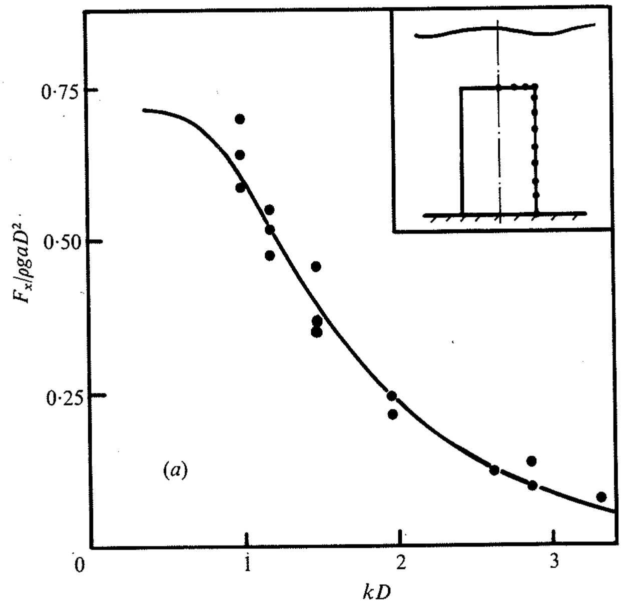
$$\mathbf{F}(t) = \mathrm{Re}\left[i\rho\sigma e^{-i\sigma t}\int_A(\phi_i+\phi_s)\,\hat{\mathbf{n}}\,dA\right],$$

in which

$$\hat{\mathbf{n}} = \mathbf{i}z'\cos\Theta + \mathbf{j}z'\sin\Theta - \mathbf{k}r', \quad dA = R\,d\Theta\,dS.$$

Substituting the Fourier expansion

$$\phi = \phi_i + \phi_s = \sum_{l=0}^{\infty}\phi_l\cos l\theta$$



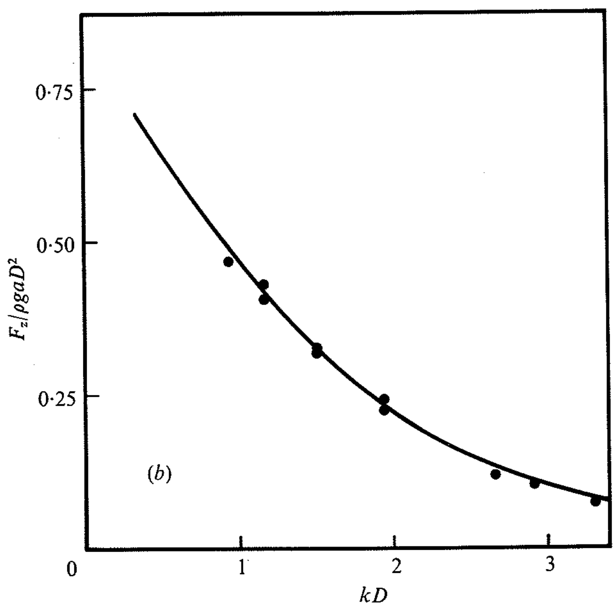


FIGURE 2. Variation of (*a*) dimensionless drag force and (*b*) dimensionless vertical force with wavenumber for a truncated circular cylinder of height 0·7 and diameter 0·4 of the water depth *h*. ●, experimental results from Hogben & Standing (1975); ——, results from the present theory using the elemental subdivision shown in the inset in (*a*). F_x = horizontal force, F_z = vertical force, ρ = fluid density, a = wave amplitude, D = cylinder diameter, k = wavenumber.

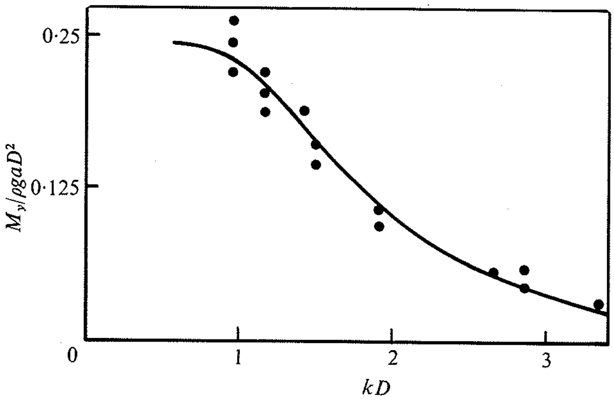


FIGURE 3. Dimensionless moment about the base of the cylinder.

and integrating with respect to θ, the orthogonality of the trigonometric terms shows that there is no transverse force, while only the $l = 0$ term contributes to vertical forces and the $l = 1$ term to horizontal forces. Thus

$$\mathbf{F}(t) = \mathrm{Re}\left[i\pi\rho\sigma\, e^{-i\sigma t}\int_S R(\mathbf{i}z'\phi_1(s) - \mathbf{j}2r'\phi_0(s))\,dS\right]$$
$$= \mathrm{Re}\left[i\pi\rho\sigma e^{-i\sigma t}\sum_{n=1}^{N} L_n R_n(\mathbf{i}z'_n\,\phi_1(s_n) - \mathbf{j}2r'_n\,\sigma\phi_0(s_n))\right]. \quad (5.6)$$

Similarly we have (1.11) for the moment on the body. Performing the manipulations we find that there is only a ('pitching') moment about the y axis, given by the $l = 1$ term:

$$\mathbf{M}(t) = \mathbf{j}\,\mathrm{Re}\left[i\pi\rho\sigma\, e^{-i\sigma t}\sum_{n=1}^{N} L_n\, r_n\, \phi_1(s_n)\,(z'_n(z_n - z_*) + r'_n\, r_n)\right],$$

where z_* is the elevation of the point about which moments are to be taken.

6. Results

A computer program was written for bodies of arbitrary cross-section. As a simple test this was applied to a right circular cylinder fixed to the bed in water of depth h: the cylinder was $0{\cdot}7h$ high and had a diameter of $0{\cdot}4h$. These are the same relative dimensions as those of one of the cylinders tested by Hogben & Standing (1975) at the National Physical Laboratory, which gave the experimental points on figures 2 and 3. Results from the calculations of the present work are shown by the continuous line in each case. Agreement for this case, as for others tested, was good. The elemental subdivision, shown in the inset of figure 2 (a), had 11 elements. Convergence of the series for the Green's function, given in the appendix, was extremely rapid, so that generally 4–6 terms, rising to 20 at the singularities, were sufficient to achieve an accuracy of 0·0001.

The computer program, using the theory and numerical procedures described in this paper, was written for axially symmetric bodies of any cross-section. Details are available from Dr L. R. Wootton, Atkins Research and Development, Epsom, Surrey.

I would like to thank Dr Wootton and colleagues for the encouragement and advice I received during the execution of this work.

Appendix

Here we give the expressions for the coefficients of the two main influence matrices described in the body of the paper. The first, a_{ij}, is the normal velocity at the point (r_i, z_i) on the body cross-section due to a unit source at the point (R_j, Z_j); similarly c_{ij} gives the induced velocity potential. Where two alternatives in curly brackets are given, the upper is used if $R_j < r_i$, the lower if $R_j > r_i$ and the mean of the two if $R_j = r_i$.

$$
\begin{aligned}
a_{ij} = kC_0 R_j L_j \cosh kZ_j \Bigg[& z_i' \cosh kz_i \left\{\begin{matrix} J_l(kR_j)\,(Y_l'(kr_i) - iJ_l'(kr_i)) \\ J_l'(kr_i)\,(Y_l(kR_j) - iJ_l(kR_j)) \end{matrix}\right\} \\
& - r_i' \sinh kz_i \left\{\begin{matrix} J_l(kR_j)\,(Y_l(kr_i) - iJ_l(kr_i)) \\ J_l(kr_i)\,(Y_l(kR_j) - iJ_l(kR_j)) \end{matrix}\right\}\Bigg] \\
+ \sum_{m=1}^{\infty} \Bigg[& 4\mu_m C_m R_j L_j \cos \mu_m Z_j \left\{\begin{matrix} I_l(\mu_m R_j) \\ K_l(\mu_m r_i) \end{matrix}\right\} \left(z_i' \cos \mu_m z_i \left\{\begin{matrix} K_l'(\mu_m r_i) \\ I_l(\mu_m R_j) \end{matrix}\right\} \right. \\
& \left. + r_i' \sin \mu_m z_i \left\{\begin{matrix} K_l(\mu_m r_i) \\ I_l(\mu_m R_j) \end{matrix}\right\}\right) \\
& + \frac{2L_j}{h} (R_j/r_i)^{\frac{1}{2}} \exp(-m\pi|R_j - r_i|/h) \left(\cos\frac{m\pi Z_j}{h} \cos\frac{m\pi z_i}{h} \right. \\
& \times \left(\left\{\begin{matrix} + \\ - \end{matrix}\right\} z_i' + \frac{\nu(r_i - R_j)}{m\pi} z_i' - \frac{z_i' h}{R_j} \frac{4l^2 - 1}{8m\pi} + \frac{z_i' h}{r_i} \frac{4l^2 + 3}{8m\pi} + \frac{\nu z_i r_i'}{m\pi} \right) \\
& + \sin\frac{m\pi Z_j}{h} \cos\frac{m\pi z_i}{h} \left(\left\{\begin{matrix} + \\ - \end{matrix}\right\} \frac{\nu Z_j z_i'}{m\pi} \right) + \sin\frac{m\pi Z_j}{h} \sin\frac{m\pi z_i}{h} \left(-\frac{\nu Z_j z_i'}{m\pi} \right) \\
& \left. + \cos\frac{m\pi Z_j}{h} \sin\frac{m\pi z_i}{h} \left(\left\{\begin{matrix} + \\ - \end{matrix}\right\} \frac{\nu z_i z_i'}{m\pi} - r_i' - \frac{\nu r_i' |R_j - r_i|}{m\pi} + \frac{h r_i' |R_j - r_i|}{R_j r_i} \frac{4l^2 - 1}{8m\pi} \right) \right) \Bigg] \\
+ L_j \Bigg[& \frac{1}{h} \left(\frac{R_j}{r_i} \right)^{\frac{1}{2}} \exp(-\pi|R_j - r_i|/h) \\
& \times \left(\frac{\left\{\begin{matrix} - \\ + \end{matrix}\right\} z_i'[\cos\pi(Z_j + z_i)/h - \exp(-\pi|R_j - r_i|/h) + r_i' \sin\pi(Z_j + z_i)/h}{1 - 2\exp(-\pi|R_j - r_i|/h) \cos\pi(Z_j + z_i)/h + \exp(-2\pi|R_j - r_i|/h)} \right) \\
& - \frac{1}{\pi} \left(\frac{z_i'(R_j - r_i) + r_i'(Z_j + z_i - 2h)}{(R_j - r_i)^2 + (Z_j + z_i - 2h)^2} + \frac{z_i'(R_j - r_i) + r_i'(Z_j + z_i)}{(R_j - r_i)^2 + (Z_j + z_i)^2} \right) \Bigg] \\
+ L_j \Bigg[& \frac{1}{h} \left(\frac{R_j}{r_i} \right)^{\frac{1}{2}} \exp(-\pi|R_j - r_i|/h) \\
& \times \left(\frac{\left\{\begin{matrix} - \\ + \end{matrix}\right\} z_i' [\cos\pi(Z_j - z_i)/h - \exp(-\pi|R_j - r_i|/h)] - r_i' \sin\pi(Z_j - z_i)/h}{1 - 2\exp(-\pi|R_j - r_i|/h) \cos\pi(Z_j - z_i)/h + \exp(-2\pi|R_j - r_i|/h)} \right) \\
& + \frac{1}{\pi} \left(\frac{r_i'(Z_j - z_i) - z_i'(R_j - r_i)}{(R_j - r_i)^2 + (Z_j - z_i)^2} \right) \Bigg]
\end{aligned}
$$

$$+\frac{L_j}{h}\left(\frac{R_j}{r_i}\right)^{\frac{1}{2}}\left(\frac{\nu z_i'}{2\pi}(r_i-R_j)-\frac{4l^2-1}{16\pi}\frac{hz_i'}{R_j}+\frac{4l^2+3}{16\pi}\frac{hz_i'}{r_i}+\frac{\nu r_i'}{2\pi}(Z_j+z_i)\right)$$

$$\times\ln\left[1-2\exp\left(-\pi|R_j-r_i|/h\right)\cos\pi\left(Z_j+z_i\right)/h+\exp\left(-2\pi|R_j-r_i|/h\right)\right]$$

$$-\frac{L_j}{\pi}\left[\frac{z_i'}{4r_i}\ln\frac{\pi^2}{h^2}((R_j-r_i)^2+(Z_j+z_i)^2)+\left(\frac{z_i'}{4r_i}+\nu r_i'\right)\right.$$

$$\left.\times\ln\frac{\pi^2}{h^2}(R_j-r_i)^2+(Z_j+z_i-2h)^2)\right]$$

$$+\frac{L_j}{h}\left(\frac{R_j}{r_i}\right)^{\frac{1}{2}}\left(\frac{\nu z_i'}{2\pi}(r_i-R_j)-\frac{4l^2-1}{16\pi}\frac{hz_i'}{R_j}+\frac{4l^2+3}{16\pi}\frac{hz_i'}{r_i}+\frac{\nu r_i'}{2\pi}(Z_j-z_i)\right)$$

$$\times\ln\left[(1-2\exp\left(-\pi|R_j-r_i|/h\right)\cos\pi(Z_j-z_i)/h+\exp\left(-2\pi|R_j-r_i|/h\right)\right]$$

$$-\frac{L_jz_i'}{4\pi r_i}\ln\frac{\pi^2}{h^2}((R_j-r_i)^2+(Z_j-z_i)^2)+\frac{L_j}{h}\left(\frac{R_j}{r_i}\right)^{\frac{1}{2}}\left[\begin{Bmatrix}-\\+\end{Bmatrix}\frac{\nu z_i'}{\pi}(Z_j+z_i)\right.$$

$$\left.+\frac{\nu r_i'}{\pi}|R_j-r_i|\begin{Bmatrix}+\\-\end{Bmatrix}\frac{4l^2-1}{8\pi}hr_i'\left(\frac{1}{r_i}-\frac{1}{R_j}\right)\right]$$

$$\times\tan^{-1}\left[\frac{\exp\left(-\pi|R_j-r_i|/h\right)\sin\pi(Z_j+z_i)/h}{1-\exp\left(-\pi|R_j-r_i|/h\right)\cos\pi(Z_j+z_i)/h}\right]$$

$$+\frac{L_j}{h}\left(\frac{R_j}{r_i}\right)^{\frac{1}{2}}\left(\begin{Bmatrix}-\\+\end{Bmatrix}\frac{\nu z_i'}{\pi}(Z_j-z_i)-\frac{\nu r_i'}{\pi}|R_j-r_i|\begin{Bmatrix}-\\+\end{Bmatrix}\frac{4l^2-1}{8\pi}hr_i'\left(\frac{1}{r_i}-\frac{1}{R_j}\right)\right)$$

$$\times\tan^{-1}\left[\frac{\exp\left(-\pi|R_j-r_i|/h\right)\sin\pi\left(Z_j-z_i\right)/h}{1-\exp\left(-\pi|R_j-r_i|/h\right)\cos\pi(Z_j-z_i)/h}\right]$$

$$+\pi^{-1}(I_1(+1,z_i)+I_1(+1,z_i-2h)+I_1(-1,-z_i))$$

$$+\frac{z_i'}{4r_i}(I_2(z_i)+I_2(z_i-2h)+I_2(-z_i))+\nu r_i'I_2(z_i-2h),$$

where

$$I_1(\epsilon,\delta)=(z_i'z_j'-\epsilon r_i'r_j')\tan^{-1}\left[\frac{r_j'(R-r_i)+z_j'(Z+\delta)}{z_j'(R_j-r_i)-r_j'(Z_j+\delta)}\right]$$

$$\left.+\tfrac{1}{2}(z_i'r_j'+\epsilon r_i'z_j')\ln\left((R-r_i)^2+(Z+\delta)^2\right)\right|_{R_1,\,Z_1}^{R_2,\,Z_2},$$

$$I_2(\delta)=L_j\left(\ln\frac{\pi^2}{h^2}-2\right)+2(z_j'(R_j-r_i)-r_j'(Z_j+\delta))\tan^{-1}\left[\frac{r_j'(R-r_i)+z_j'(Z+\delta)}{z_j'(R_j-r_i)-r_j'(Z_j+\delta)}\right]$$

$$\left.+(r_j'(R-r_i)+z_j'(Z+\delta)\ln\left[(R-r_i)^2+(Z+\delta)^2\right]\right|_{R_1,\,Z_1}^{R_2,\,Z_2},$$

in which the limits are

$$R_1=R_j-\tfrac{1}{2}r_j'L_j,\quad R_2=R_j+\tfrac{1}{2}r_j'L_j,\quad Z_1=Z_j-\tfrac{1}{2}z_j'L_j,\quad Z_2=Z_j+\tfrac{1}{2}z_j'L_j.$$

$$c_{ij}=L_jR_jC_0\cosh(kz_i)\cosh(kZ_j)J_l\left(k\begin{Bmatrix}R_j\\r_i\end{Bmatrix}\right)\left(Y_l\left(k\begin{Bmatrix}r_i\\R_j\end{Bmatrix}\right)-iJ_l\left(k\begin{Bmatrix}r_i\\R_j\end{Bmatrix}\right)\right)$$

$$+\sum_{m=1}^{\infty}\left[4L_jR_jC_m\cos(\mu_mz_i)\cos(\mu_mZ_j)K_l\left(\mu_m\begin{Bmatrix}r_i\\R_j\end{Bmatrix}\right)I_l\left(\mu_m\begin{Bmatrix}R_j\\r_i\end{Bmatrix}\right)\right.$$

$$\left.-\frac{2L_j}{m\pi}\left(\frac{R_j}{r_i}\right)^{\frac{1}{2}}\exp\left(-m\pi|R_j-r_i|/h\right)\cos m\pi Z_j\cos m\pi z_i\right]$$

$$+\frac{L_j}{2\pi}\left[\ln\frac{\pi^2}{h^2}((R_j-r_i)^2+(Z_j+z_i)^2)+\ln\frac{\pi^2}{h^2}((R_j-r_i)^2+(Z_j+z_i-2h)^2)\right.$$
$$\left.+\ln\frac{\pi^2}{h^2}((R_j-r_i)^2+(Z_j-z_i)^2)\right]$$
$$-\frac{L_j}{2\pi}\left(\frac{R_j}{r_i}\right)^{\frac{1}{2}}\ln\left[1-2\exp\left(-\pi|R_j-r_i|/h\right)\cos\pi(Z_j+z_i)/h+\exp\left(-2\pi|R_j-r_i|/h\right)\right]$$
$$-\frac{L_j}{2\pi}\left(\frac{R_j}{r_i}\right)^{\frac{1}{2}}\ln\left[1-2\exp\left(-\pi|R_j-r_i|/h\right)\cos\pi(Z_j-z_i)/h+\exp\left(-2\pi|R_j-r_i|/h\right)\right]$$
$$-\frac{1}{2\pi}\left[I_2(z_i)+I_2(z_i-2h)+I_2(-z_i)\right].$$

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