CHAPTER 2

ACCURATE NUMERICAL SOLUTIONS FOR NONLINEAR WAVES

by

J.D. Fenton and M.M. Rienecker
School of Mathematics
University of New South Wales
Kensington, N.S.W., Australia 2033

This paper describes numerical methods for the accurate solution of the nonlinear equations for water waves propagating on irrotational flow over a horizontal bed. Fourier approximation is used throughout. Firstly, the problem of waves propagating without change is considered, giving a set of nonlinear equations which may be conveniently solved by Newton's method. It is emphasized that the usual specification of water depth, wave height and wave period is not enough to solve the problem - an assumption as to wave speed or mean current or mass flux must be included. Comparing results with previous theoretical and experimental results, good agreement was obtained. In the second part unsteady wave motion is examined, and a numerical method proposed for studying the evolution of unsteady disturbances. This is applied to the case of a solitary wave being reflected by a vertical wall. Close agreement with experimental results is obtained. In addition, design criteria for force and moment on the wall are suggested.

1. INTRODUCTION

In water wave problems where viscous forces may be assumed to be negligible and the fluid initially irrotational, fluid motion is governed by Laplace's equation for a velocity potential, and for two-dimensional flow, a stream function as well. This equation is linear, but for water wave problems must be solved subject to nonlinear boundary conditions on the free surface, which is also an unknown of the problem, in addition to the relatively simple linear boundary conditions on solid boundaries. It is the nonlinear dynamic and kinematic conditions which make solution difficult.

Approximate solutions of the equations have generally been found either by assuming that the waves are of small amplitude and nonlinearities ignored, or by assuming that the water depth is small relative to the wavelength, giving the long wave equations, or by a combination of both. These approximations can describe most phenomena associated with the propagation of waves to a first level of approximation. However, in some engineering and experimental applications it is necessary to have more accurate solutions of the complete set of nonlinear equations; for example, the design of maritime structures necessitates an accurate knowledge of the fluid velocity and pressure fields acting on them in the presence of waves.

This paper describes the development of numerical methods for the solution of the exact nonlinear equations for waves on fluid in which the flow is irrotational. In §2 the problem of waves propagating without change over water of constant depth is considered. The two best known analytical approximations to this problem are (i) Stokes' solution, in which series are essentially obtained in terms of powers of wave steepness (height/wavelength) and (ii) the cnoidal solution, using series in terms of shallowness (water depth/wavelength) which may be recast to include wave height. Each of these approaches breaks down for opposite extremes of depth - Stokes wave solutions are not valid for shallow water while cnoidal wave solutions are not valid for deep water. For high waves, neither is particularly accurate without the inclusion of many terms and the use of series enhancement techniques (see Cokelet (1977) for Stokes , and Fenton (1979) for cnoidal wave solutions). A method which has the potential for describing even very high waves in water of almost all depths is that which is known as the "stream function" method (Dean, 1965), but which does not depend on the stream function as such; rather it is a method of Fourier approximation of the stream function, and it is the Fourier approach wherein its power lies. The original technique has been modified by other workers, however, it still seems rather more complicated to use than need be the case. An alternative Fourier approximation method is presented in §2.

The rather more general problem of the complete unsteady equations for arbitrary disturbances is tackled in §3. This problem has received little attention except for the marker and cell method of Chan & Street (1970), the nonlinear integral equation of Byatt-Smith (1971) and the integro-differential equations of Longuet-Higgins and Cokelet (1976) for waves on infinitely deep water. Here, a new method to follow the time evolution of an initial disturbance is proposed wherein all dependent variables are accurately approximated by finite Fourier series. The only approximation is the truncation of the series. This method is applicable to water of any (variable) depth, however all motions and depth variation must be horizontally periodic. The method is applied to the problem of a long, but finite, wave approximating a solitary wave moving over a horizontal bottom and being reflected by a vertical wall. In previous approaches to this problem (for example Chan & Street (1970)) values for the force and moment on the wall due to the wave impact have not been given even though these are two very important engineering quantities. The maximum values for the force and moment are obtained from the impact of a solitary wave, hence any results for these quantities should provide the design criteria for a vertical wall subject to wave impact. In §3 results are given for these and also for the wave run-up which is found to agree closely with experimental results.

2. STEADY WAVES

2.1 Formulation of equations

The problem considered is that of two-dimensional periodic waves propagating without change of form over a layer of fluid on a horizontal bed. With horizontal co-ordinate x and vertical co-ordinate y, the origin is on the bed, and moves in the x direction with the same speed as the waves so that all motion is steady in this frame of reference. If the fluid is incompressible a stream function $\psi(x,y)$ exists such that the velocity components (u,v) are given by

$$u = \frac{\partial\psi}{\partial y} \text{ and } v = -\frac{\partial\psi}{\partial x},$$

and if the motion is irrotational, ψ satisfies Laplace's equation throughout the fluid:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0. \tag{2.1}$$

The boundary conditions to be satisfied are

$$\psi(x,0) = 0 \tag{2.2}$$

$$\psi(x,\eta(x)) = -Q \tag{2.3}$$

where $y = \eta(x)$ on the free surface and Q is a positive constant denoting the total volume rate of flow underneath the steady wave per unit length in a direction normal to the (x,y) plane. With this sign convention the apparent flow under the stationary wave is from right to left, in the negative x-direction. On the free surface, the pressure is constant so that Bernoulli's equation gives

$$\tfrac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] + \eta = R, \tag{2.4}$$

where R is the Bernoulli constant. In these equations all variables have been non-dimensionalised with respect to the average depth $\bar{\eta}$ and gravitational acceleration g : that is, x is used for $x/\bar{\eta}$, y for $y/\bar{\eta}$, η for $\eta/\bar{\eta}$, ψ for $\psi/(g\bar{\eta}^3)^{\frac{1}{2}}$, Q for $Q/(g\bar{\eta}^3)^{\frac{1}{2}}$ and R for $R/g\bar{\eta}$. Other dimensionless variables to be introduced are – wave speed: c for $c/(g\bar{\eta})^{\frac{1}{2}}$, wavenumber: k for $k\bar{\eta} = 2\pi\bar{\eta}/\lambda$ where λ is the wavelength, wave period: τ for $\tau(g/\bar{\eta})^{\frac{1}{2}}$, and an arbitrary reference level: D for $D/\bar{\eta}$.

If the symmetry of the wave about the crest is exploited, a series for $\psi(x,y)$ can be written

$$\psi(x,y) = B_o y + \sum_{j=1}^{N} B_j \frac{\sinh jky}{\cosh jkD} \cos jkx, \tag{2.5}$$

satisfying (2.1) and (2.2). The $B_o,\ldots B_N$ are constant for a particular wave. The assumption that N is finite, for computational purposes, is the only approximation made in this method of solution. This assumed form is similar to Dean's (1965) series. The two differences are:

(i) the leading coefficient B_o is not necessarily the wave speed, which will be discussed below, and (ii) the inclusion of coshjkD in the denominator so that waves in deep water can be studied. Without this factor the B_j need to become exponentially small for convergence since the sinhjky go exponentially large. In subsequent iterative solution methods for the equations this cannot be guaranteed and the method may be rather unstable. However, sinhjky/coshjkD $\sim \exp(|j|k(y-D))$ near the surface, so if D is greater than the crest height this ratio is always less than 1 and no problems of exponential magnification arise.

From here, the present method and the "stream function" method have very few similarities. In the latter, equations are obtained by requiring the mean-square error in one free surface boundary condition to be a minimum. Here (2.5) is used to satisfy both nonlinear boundary conditions (2.3) and (2.4) at a finite number of equally-spaced points, giving a number of nonlinear equations which can be directly solved using standard techniques.

To solve the problem numerically the equations produced by the substitution of (2.5) into (2.3) and (2.4), that is

$$B_o\eta + \sum_{j=1}^{N} B_j \frac{\sinh jk\eta}{\cosh jkD} \cos jkx = -Q, \tag{2.6}$$

and

$$\tfrac{1}{2}\{B_o + k\sum_{j=1}^{N} jB_j \frac{\cosh jk\eta}{\cosh jkD} \cos jkx\}^2 + \tfrac{1}{2}\{k\sum_{j=1}^{N} jB_j \frac{\sinh jk\eta}{\cosh jkD} \sin jkx\}^2 + \eta = R, \tag{2.7}$$

which are valid for all x, are to be satisfied at 2N points equally spaced over one wavelength, though by symmetry only N+1 points, from the wave crest to the trough, need to be considered. In the development of this method, tests were run where the computational points were clustered near the crest and sparsely spaced in the long flat trough, however it was found that there was very little gain in accuracy, and it is recommended that equi-spaced points be used in applications.

Let $\eta_m = \eta(x_m)$, where $x_m = m\lambda/2N$, $m = 0,1,\ldots N$, and $kx_m = m\pi/N$, so that (2.6) and (2.7) become

$$B_o\eta_m + \sum_{j=1}^{N} B_j \frac{\sinh jk\eta_m}{\cosh jkD} \cos(jm\pi/N) + Q = 0, \tag{2.8}$$

$$\tfrac{1}{2} u_m^2 + \tfrac{1}{2} v_m^2 + \eta_m - R = 0, \tag{2.9}$$

also for $m = 0,1,\ldots N$, where

$$u_m = u(x_m, \eta_m) = B_o + k\sum_{j=1}^{N} jB_j \frac{\cosh jk\eta_m}{\cosh jkD} \cos(jm\pi/N)$$

$$\text{and} \quad v_m = v(x_m, \eta_m) = k \sum_{j=1}^{N} jB_j \frac{\sinh jk\eta_m}{\cosh jkD} \sin(jm\pi/N).$$

These 2N + 2 nonlinear equations involve the 2N + 5 unknowns $\eta_m(m = 0,\ldots,N)$, $B_j(j = 0,\ldots,N)$, k, Q and R. To obtain a solution 3 more equations must be specified. Since variables have been non-dimensionalised with respect to $\bar{\eta}$, the mean depth is 1. The simple trapezoidal rule can be used to give one more equation:

$$\frac{1}{2N}[\eta_o + \eta_N + 2\sum_{j=1}^{N-1} \eta_j] - 1 = 0, \tag{2.10}$$

where this can be shown to have an error proportional to the (N+1)th term of a Fourier series for η. This is the same level of approximation as the Fourier expansion for ψ which was truncated after the Nth term.

Now, numerical values for <u>any</u> two of the variables can be specified and the equations solved. However, for practical problems it is usually values of H (for $H/\bar{\eta}$), the wave height and τ (for $\tau(g/\bar{\eta})^{\frac{1}{2}}$) the wave period which define the problem. Two additional equations which specify these physical parameters are (i) the definition of H as being the elevation difference between crest and trough:

$$\eta_o - \eta_N - H = 0, \tag{2.11}$$

where η_o is the surface elevation at the crest and η_N that at the trough, and (ii) the statement that wave period is equal to the wave length divided by the wave speed, or

$$kc\tau - 2\pi = 0, \tag{2.12}$$

where the introduction of the wave speed c has added one more variable. It is possible to solve the problem in a frame relative to which motion is steady, without having to define the wave speed, by specifying the wavelength $\lambda(= 2\pi/k)$ instead of wave period. However, in most situations the waves are viewed from a different frame of reference in which they are not stationary and the wave period in this frame is measured. Hence an assumption as to the speed at which the waves travel must be included. The waves could travel at <u>any</u> speed, in a given frame, without change of form, so that the specification of the usual three quantities, water depth, wave height and period is not sufficient to define the problem. Either the wave speed must be specified or a value given (measured or assumed) for a quantity which determines the wave speed; for example, the mean current C_E (the mean Eulerian velocity throughout the fluid), or the mean Stokes drift C_s (the mass transport velocity) may be specified. In the steady frame, the mean velocity on any level over one wavelength is B_o, so that in a frame through which the waves pass at speed c, the time mean velocity at all points always within the fluid $C_E = c + B_o$ and so the appropriate equation to use if C_E is specified is

$$c + B_o - C_E = 0. \qquad (2.13a)$$

Some previous works have allowed for the specification of C_E, for example Dean (1965), however many papers concerned with the application of wave theory have implicitly assumed $C_E = 0$, for example Skjelbreia & Hendrickson (1961) and Fenton (1979). In many situations however, the mean current is not as fundamental a quantity in determining the wave speed as is the rate of mass transport: for example, wave tank experiments with a closed end must have no net flux over any section. In the steady wave, a volume rate per unit span of Q is passing under the surface, but from right to left with the convention chosen above, so that the mean velocity of all the fluid particles is $-Q/\bar{\eta} = -Q$, because $\bar{\eta}$ is unity in the present non-dimensionalisation. Thus, if C_s is the mass-transport velocity in the stationary frame $C_s = c - Q$, giving the appropriate equation to use if C_s is specified:

$$c - Q - C_s = 0. \qquad (2.13b)$$

2.2. Solution of nonlinear equations by Newton's method

The system of nonlinear equations (2.8 -.13) may be written

$$f_i(\eta_j, B_j (j = 0,\ldots N),\ c,k,Q,R) = 0, \qquad (2.14)$$

$i = 1, \ldots 2N + 6$, where for $i = 1$ to $(N+1)$ the f_i represent (2.8), for $i = (N+2)$ to $(2N+2)$ the f_i represent (2.9), f_{2N+3} is (2.10), f_{2N+4} is (2.11), f_{2N+5} is (2.12) and f_{2N+6} is either (2.13a) or (2.13b). This set of equations can be solved by Newton's method which iterates with quadratic convergence to a solution from an initial approximation. If the system of equations (2.14) is represented by

$$f_i(Z_\ell; \ell = 1,\ldots,2N+6) = 0, \quad i = 1,\ldots,2N+6 \qquad (2.15)$$

where the Z_ℓ are the arguments of f_i as in (2.14), and if the approximate solution after the nth iteration is Z_ℓ^n, $\ell = 1,\ldots,(2N+6)$ the error in each equation is $f_i(Z_\ell^n;\ \ell = 1,\ldots 2N+6) = f_i^n$. From a multivariable Taylor series expansion the error at the next iteration is

$$f_i^{n+1} = f_i^n + \sum_{\ell=1}^{2N+6} \left(\frac{\partial f_i}{\partial Z_\ell}\right)^n \left(Z_\ell^{n+1} - Z_\ell^n\right) + \ldots\ldots, \quad i = 1,\ldots 2N+6,$$

where $$\left(\frac{\partial f_i}{\partial Z_\ell}\right)^n = \frac{\partial f_i}{\partial Z_\ell}\,(Z_j^n,\ j = 1,\ldots,2N+6).$$

However, the desired result is $f_i^{n+1} = 0$ for each $i = 1,\ldots 2N+6$, and the solution Z_ℓ^{n+1}, $\ell = 1,\ldots 2N+6$ which approximately yields this result is found by truncating the series after the terms shown and solving the resulting system of linear simultaneous equations written as a matrix equation:

$$\left[\left(\frac{\partial f_i}{\partial Z_\ell}\right)^n\right][Z_\ell^{n+1} - Z_\ell^n] = -[f_i^n]\ .$$

The derivatives $\frac{\partial f_i}{\partial Z_\ell}$ have convenient analytical expressions obtained by differentiation of the equations (2.8 -.13). For example, from (2.8), $\partial f_i/\partial B_o = \eta_m$, and so on. Expressions for all these derivatives are simply obtained and are given in Rienecker and Fenton (1980).

The initial approximation to the solution is assumed to be a linear sinusoidal wave, that is, $\eta_m = 1 + \frac{1}{2} H \cos m\pi/N$, $m = 0,\ldots N$; $B_o = -c$; $B_1 = -\frac{1}{4} H/ck$; $B_j = 0$, $j = 2,\ldots N$; $R = 1 + \frac{1}{2}c^2$; $Q = c$, where c and k are found recursively from

$$k = 2\pi/\tau c$$

$$c = (\tanh k/k)^{\frac{1}{2}},$$

with an initial guess $c = 1$ corresponding to a long wave approximation.

2.3. Results

The method was run on a CDC Cyber 171 computer; convergence of the iteration was extremely rapid so that 5 iterations were usually sufficient to obtain convergence to 12 decimal places. For very high waves it was found that the linear approximation was not a sufficiently accurate initial estimate of the solution and the method did not converge. In this situation it was necessary to extrapolate the initial estimate from converged solutions for lower waves.

With this solution for the elevation of points on the free surface η_m, for the Fourier coefficients B_j, and for c,k,Q and R, it is possible to obtain results for other quantities which may be of interest, in particular the fluid velocities and pressure at any point. All physical variables considered so far have been in a co-ordinate system which moves with the wave so that all motion is steady in that frame. The steady velocities at any point are from differentiation of (2.5):

$$u(x,y) = B_o + k \sum_{j=1}^{N} u_j(x,y), \text{ and}$$

$$v(x,y) = k \sum_{j=1}^{N} v_j(x,y),$$

where

$$u_j(x,y) = jB_j \frac{\cosh jky}{\cosh jkD} \cos jkx, \quad \text{and} \tag{2.16a}$$

$$v_j(x,y) = jB_j \frac{\sinh jky}{\cosh jkD} \sin jkx. \tag{2.16b}$$

The pressure at any point, $p(x,y)$, representing $p(x,y)/\rho g\bar{\eta}$ where ρ is fluid density, is given by

$$p(x,y) = R - y - \tfrac{1}{2}[u^2(x,y) + v^2(x,y)].$$

Now consider a stationary co-ordinate system (X,Y) on the bed in which motion is unsteady as the waves move from left to right with speed c. If the wave crest is at $X = X_o$ when $t = 0$, the unsteady fluid velocities $U(X,Y,t)$ and $V(X,Y,t)$ become

$$U(X,Y,t) = c + B_o + k \sum_{j=1}^{N} u_j(X-X_o-ct,Y)$$

$$V(X,Y,t) = k \sum_{j=1}^{N} v_j(X-X_o-ct,Y),$$

where the (u_j,v_j) are defined in (2.16a,b). Unsteady pressure is given by

$$p(X,Y,t) = R - Y - \tfrac{1}{2}[u^2(X-X_o-ct,Y) + v^2(X-X_o-ct,Y)].$$

In addition to the above local quantities, gross integral quantities of the wave train such as potential energy may be calculated from the converged solution. Expressions for these are given in Rienecker and Fenton (1980).

2.4. Accuracy

Cokelet (1977) presented a number of accurate results for integral quantities of the wave train. The wave speed c, for the case $C_E = 0$, was used to test the accuracy of the present method. For each of four different values of kQ/c, which is a measure of the ratio of wavelength to depth of fluid, the variation of c^2 with H is shown on Fig. 1. The results indicate close agreement with Cokelet. Only for the very highest and longest wave with relatively coarse numerical approximation (small N) were significant errors obtained. In these cases it was not the height of the wave which caused the lack of accuracy as the method makes no approximation as to height. Instead, for higher waves the crest becomes increasingly sharp and the Fourier approximation becomes less accurate. Also, for longer waves where the wave is more like an isolated hump on otherwise undisturbed fluid, larger values of N are required to describe the wave accurately.

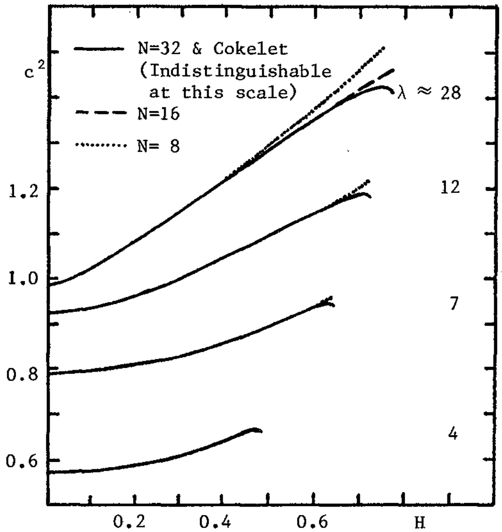

Figure 1. Comparison between the present method and the results of Cokelet (1977) for wave speed squared, c^2, as a function of wave height H. Each curve is drawn for a constant value of kQ/c taken from Cokelet's tables A2, A4, A6, A8, giving the almost constant values of wavelength shown.

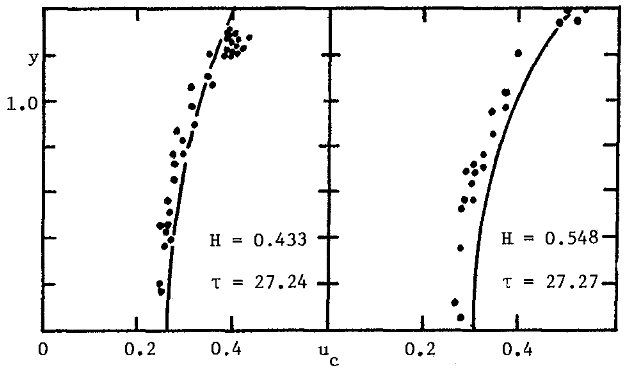

Figure 2. Horizontal fluid velocity under the wave crest, u_c, plotted against height y above the bottom: comparison between present method (——) and the experimental results of Le Méhauté et al (1968). Dimensionless wave heights and periods are shown.

To compare the predictions of the Fourier method for fluid velocities with experiment, the results of Le Méhauté, Divoky and Lin (1968) were used. As the experiments were performed in a closed wave-tank, the condition (2.13b) was used to determine the wave speed $c = Q$. This condition does not seem to have been used in previous comparisons with these experiments, where the incorrect assumption $C_E = 0$ has been used. Also, it should be pointed out that the experimental velocities measured were particle velocities averaged over a finite time and not the instantaneous velocities given by all wave theories. It can be shown that these experimental velocities should be less than those predicted. In Fig. 2, two sets of experimental results are shown for the two longest waves generated, with a wavelength of about 30 times the depth in each case. These longest waves should provide a severe test for the Fourier approximation method, as described above. It can be seen that agreement between theory and experiment was quite close.

3. UNSTEADY WAVES

3.1. Equations

A solution is sought to the equations governing the evolution of waves travelling over a layer of fluid on a horizontal bed. If the fluid is irrotational a velocity potential $\phi(x,y,t)$ exists such that velocity components $u(x,y,t)$ and $v(x,y,t)$ parallel to the x and y co-ordinates respectively are given by

$$u = \partial\phi/\partial x \text{ and } v = \partial\phi/\partial y.$$

If the motion is incompressible, ϕ satisfies Laplace's equation throughout the fluid:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{3.1}$$

Since the velocity normal to the bed is zero,

$$\frac{\partial\phi}{\partial y}(x,0,t) = 0, \tag{3.2}$$

where the co-ordinate origin has been placed on the horizontal bed. The case where the bed is not horizontal can be treated by the methods of this section, and is described in Fenton & Rienecker (Manuscript in preparation).

On the free surface $y = \eta(x,t)$, the kinematic and dynamic boundary conditions must be satisfied:

$$\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} - \frac{\partial\phi}{\partial y} = 0 \tag{3.3}$$

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] + \eta = C, \tag{3.4}$$

where C is a constant. In these equations all variables have been nondimensionalized with respect to the mean depth $\bar{\eta}$, and gravitational acceleration g. If a simple method can be found for solving the linear equations (3.1) and (3.2), and if a convenient and accurate method can be found for approximating all the space derivatives in (3.3) and (3.4), then subject to known initial conditions, (3.3) and (3.4) can be used to advance the solution in time, so that the evolution of disturbances can be followed.

3.2. Numerical approximation and solution of equations

The Fourier methods of §2 are capable of great accuracy, and similar approximations will be used here for $\phi(x,y,t)$ and also for $\eta(x,t)$ so that term by term differentiation can be used for derivatives of each. It is assumed that the dependent variables ϕ and η can be represented by Fourier series in x throughout the region of interest, implying that all motion is periodic in x with some finite period L, referred to as the wavelength.

An expansion for $\phi(x,y,t)$ may be written

$$\phi(x,y,t) = \alpha x + \frac{1}{N} \sum_{j=-\frac{N}{2}+1}^{N/2} A_j(t) \frac{\cosh jky}{\cosh jkD} \exp(-ijkx) \tag{3.5}$$

for some finite value of N, where $k = 2\pi/L$. In general, ϕ is not periodic but will change by a finite amount over the region of interest. The leading term αx has been introduced to allow for the discontinuity in ϕ over one wavelength where $\alpha = (\phi(L,y,t) - \phi(0,y,t))/L$. Now, the finite complex Fourier series shown has to approximate the function $\phi(x,y,t) - \alpha x$ which is continuous. In this case the Fourier series converges absolutely and can be differentiated term by term. The Fourier coefficients $A_j(t)$ are complex, while the $\cosh jkD$ in the denominator (D is an arbitrary depth scale, conveniently chosen to be the mean) has been introduced so that the method may also be used for deep water, as for the steady wave expansion in §2.

A finite complex Fourier series can also be written for $\eta(x,t)$:

$$\eta(x,t) = \frac{1}{N} \sum_{j=-\frac{N}{2}+1}^{N/2} Y_j(t) \exp(-ijkx). \tag{3.6}$$

If η is known at the N discrete points $x_m = mL/N, m = 0....N-1$, such that

$$\eta_m = \eta(x_m,t) = \frac{1}{N} \sum_{j=-\frac{N}{2}+1}^{N/2} Y_j(t) \exp(-i2\pi mj/N), \tag{3.7}$$

then it can be shown that the Y_j can be simply obtained from the η_m:

$$Y_j = \sum_{m=0}^{N-1} \eta_m \exp(i2\pi mj/N), \quad j = -\frac{N}{2}+1,\ldots,\frac{N}{2}.$$

In this form the Y_j are said to be obtained from the η_m by a Discrete Fourier Transform,

$$Y_j = \mathcal{D}[\eta_m],$$

and (3.7) is termed an Inverse Discrete Fourier Transform,

$$\eta_m = \mathcal{D}^{-1}[Y_j].$$

Similarly, $$\phi(x_m,y,t) = \alpha x_m + \mathcal{D}^{-1}[A_j(t)\,\frac{\cosh jky}{\cosh jkD}]$$

and $$A_j(t)\,\frac{\cosh jky}{\cosh jkD} = \mathcal{D}[\phi(x_m,y,t) - \alpha x_m].$$

Now, the series (3.5) and (3.6) can be differentiated term by term, and substituting $x = x_m$ series (3.8) and (3.9) are obtained, where a subscript x,y or t denotes partial differentiation with respect to that variable. These expressions should be very accurate, provided the coefficients A_j and Y_j decay sufficiently quickly as $|j| \to N/2$, where the series are truncated.

$$\phi_x(x_m,y,t) = \alpha - ik\mathcal{D}^{-1}[jA_j(t)\,\frac{\cosh jky}{\cosh jkD}] \tag{3.8}$$

$$\phi_y(x_m,y,t) = k\mathcal{D}^{-1}[jA_j(t)\,\frac{\sinh jky}{\cosh jkD}]\,. \tag{3.9}$$

Further differentiation of (3.5) shows that Laplace's equation (3.1) is identically satisfied by (3.5), as is the bottom boundary condition (3.2). Also,

$$\eta_x(x_m,t) = -ik\mathcal{D}^{-1}[jY_j(t)] = -ik\mathcal{D}^{-1}[j\mathcal{D}[\eta_n]].$$

The remaining equations, the nonlinear free surface conditions (3.3-.4) become:

$$\begin{aligned}\frac{\partial\phi}{\partial t}(x_m,\eta_m,t) = C - \eta_m &- \tfrac{1}{2}\{\alpha - ik\mathcal{D}^{-1}[jA_j(t)\,\frac{\cosh jk\eta_m}{\cosh jkD}]\}^2\\ &- \tfrac{1}{2}k^2\{\mathcal{D}^{-1}[jA_j(t)\,\frac{\sinh jk\eta_m}{\cosh jkD}]\}^2, \text{ and}\end{aligned} \tag{3.10}$$

$$\begin{aligned}\frac{\partial\eta}{\partial t}(x_m,\eta_m,t) = k\mathcal{D}^{-1}&[jA_j(t)\,\frac{\sinh jk\eta_m}{\cosh jkD}]\\ &+ ik[\alpha - ik\mathcal{D}^{-1}[jA_j(t)\,\frac{\cosh jk\eta_m}{\cosh jkD}]]\mathcal{D}^{-1}[j\mathcal{D}[\eta_n]],\end{aligned} \tag{3.11}$$

each for $m = 0,\ldots,N-1$. Clearly, if F and G represent the right sides of these equations, then (3.10) and (3.11) may be written for given k, α, C and D:

$$\frac{\partial\phi}{\partial t}(x_m,\eta_m,t) = F\{A_j(t),\ j = -\frac{N}{2}+1,\ldots,\frac{N}{2};\ \eta(x_n,t), n=0,\ldots,N-1\},$$

and
$$\frac{\partial\eta}{\partial t}(x_m,t) = G\{A_j(t),\ j = -\frac{N}{2}+1,\ldots,\frac{N}{2};\ \eta(x_n,t),\ n = 0,\ldots,N-1\},$$

for $m = 0,\ldots,N-1$. If all values of $A_j(t)$ and $\eta(x_n,t)$ are known at any time t, then $\partial\phi/\partial t$ and $\partial\eta/\partial t$ may be evaluated, and by discretizing the time domain, the solution may be stepped forward in time. Defining time $t_\ell = \ell\,\Delta\, t$, and using centred finite differences to approximate the time derivative, the value of η at the next time step can be written

$$\eta(x_m,t_{\ell+1}) = \eta(x_m,t_{\ell-1}) + 2\Delta t\,\frac{\partial\eta}{\partial t}(x_m,t_\ell) + O(\Delta t^3),$$

that is,

$$\eta(x_m,t_{\ell+1}) \approx \eta(x_m,t_{\ell-1}) + 2\Delta t G\{A_j(t_\ell), j = -\frac{N}{2}+1,\ldots,\frac{N}{2};\eta(x_n,t_\ell),\ n = 0,\ldots N-1\}. \quad (3.12)$$

In the same manner,

$$A_j(t_{\ell+1}) \approx A_j(t_{\ell-1}) + 2\Delta t\,\frac{\partial A_j}{\partial t}(t_\ell), \quad (3.13)$$

where $\partial A_j(t_\ell)/\partial t$ is found from the solution of the N simultaneous linear equations in N unknowns obtained from differentiation of (3.5) with respect to time:

$$\frac{\partial\phi}{\partial t}(x_m,\eta_m,t_\ell) = \frac{1}{N}\sum_{j=-\frac{N}{2}+1}^{N/2} \frac{\partial A_j}{\partial t}(t_\ell)\,\frac{\cosh jk\eta_m(t_\ell)}{\cosh jkD}\,\exp(-i2\pi mj/N) \quad (3.14)$$

for $m = 0,\ldots,N-1$. Unfortunately these equations cannot be simply inverted by Fourier means to give $\partial A_j/\partial t$ since the $\eta_m(t_\ell) = \eta(x_m,t_\ell)$ vary with m.

To commence solution, initial values of $\eta(x_m,t_\ell)$ at $\ell = 0$ and 1 must be known, as well as values of $A_j(t_\ell)$ $\ell = 0,1$. The Fourier coefficients are simply found if the values of ϕ are known on some line of constant elevation d:

$$A_j(t_\ell)\,\frac{\cosh jkd}{\cosh jkD} = \mathcal{D}[\phi(x_m,d,t_\ell) - \alpha x_m],\ \ell = 0,1.$$

With these initial values known, $\partial\phi/\partial t$ and $\partial\eta/\partial t$ on the surface can be found from (3.10) and (3.11), $\partial A_j/\partial t$ found from (3.14), and η and A_j found at the next timestep from (3. 12) and (3.13). In this way, the evolution of some known initial disturbance can be followed.

This method of time stepping is susceptible to instability in some finite difference schemes, hence it is necessary to determine if and when the method can be unstable here. To do this, the equations (3.3 & .4) were linearized, and a solution assumed. It was found that for waves of wavelength L/ν, where ν is an integer which can take any value between 1 and N/2, on a stream of velocity U, that the stability criterion is

$$\sigma(\nu)\Delta t \leqslant 1,$$

where $\sigma(\nu)$ is the apparent radian frequency of waves of length L/ν:

$$\sigma(\nu) = \nu kU \pm (\nu k \tanh \nu k)^{\frac{1}{2}}, \quad \nu = 1,\ldots,N/2$$

in which the second part is the frequency of the waves travelling on otherwise undisturbed fluid, while the term νkU is the Doppler contribution to the apparent frequency by the waves being carried along on a current U. The criterion $\sigma(\nu)\Delta t \leqslant 1$ means that time steps must be taken sufficiently small so that a minimum number of time computational points are taken in any one cycle of every frequency component. The most demanding case is the highest frequency component $\nu = N/2$. For $\nu k = Nk/2 = N\pi/L = \pi/\Delta x$ sufficiently large, the criterion is approximately

$$\Delta t \leqslant (\Delta x/\pi)^{\frac{1}{2}}/(1 + U(\pi/\Delta x)^{\frac{1}{2}}).$$

The most severe case is obtained when U is finite and the waves are convected along with the flow giving a higher apparent frequency at a point. If the fluid is quiescent, $U = 0$, the criterion becomes

$$\Delta t \leqslant (\Delta x/\pi)^{\frac{1}{2}},$$

which is a relatively weak restriction, when compared with those obtained from diffusion and other equations.

3.3. Reflection of solitary waves by a vertical wall

Here, the method developed in §3.2 is applied to the particular problem of a solitary wave propagating over water of constant depth and being reflected by a vertical wall, a situation studied by Chan and Street (1970) who developed a numerical method for free surface flows based on a marker and cell technique, using finite differences. This problem is of some importance, for the solitary wave is the fastest, highest, largest, and most energetic of all steady waves, comparing the results of Longuet-Higgins & Fenton (1974) and those of Cokelet (1977). In addition, the effects on a wall of an incident solitary wave are greater than those due to a standing wave system even of long wavelength, because the solitary wave can be of full breaking height even before it strikes the wall, and at the wall will rise to a height about twice that, exerting considerable force. The standing waves, being periodic in time, can never rise greater than their breaking height at the wall, and since this height is lower than the breaking height of the solitary wave, the latter should exert the greatest forces possible and its effects should be used as the design criteria for wall forces and moments. Unfortunately, previous studies have not reported on the forces and moments caused.

Since the horizontal fluid velocity at the wall is zero, it can be seen that the problem of a solitary wave on a wall is precisely one half of the equivalent problem of two equal waves propagating in opposite directions, provided that the effects of viscosity are negligible. Both problems have been studied experimentally by Maxworthy (1976), who reported some unusual aspects of the time lag at the wall of the wave crest.

The method described in §3.2 above was tested on the problem of two equal waves propagating in opposite directions described more fully in Fenton & Rienecker (in preparation). In the following discussion only one half of the problem will be considered and will be referred to as the reflection of a single wave by a vertical wall. As none of the methods developed in this work can handle an infinitely long region, a finite length only was considered for computational purposes. To obtain the initial conditions, where the incoming wave does not yet sense appreciably the presence of the wall, a steady wave solution of specified height and period was calculated, using the method of §2. From trial runs it was found that the resulting body of the wave was independent of wave length, if this exceeded some 20 times the water depth for low waves (height 0.1 of the depth), this limit smoothly decreasing to 10 depths for waves of height 0.4 and above. Each of these waves was considered to be a very close approximation to the solitary wave of the same height, this being verified when quantities such as the propagation speed and potential energy were compared with results for the solitary wave from Longuet-Higgins & Fenton (1974). With the steady wave solution so obtained, two such numerical waves were placed together, so that their velocities of propagation were in opposite directions and of a magnitude such that horizontal fluid velocity under the long flat trough was zero, corresponding to two solitary waves coming together over otherwise undisturbed water. Then each wave was shifted forward slightly by an amount equal to the distance travelled in one computational time step, as if the other wave did not exist (an excellent approximation to motion in this first stage in view of the flatness of the trough, zero velocity under it and the small time step used). With these first two positions of each wave, two complete sets of initial conditions were calculated, so that the leapfrog timestepping procedure described in §3.2 could be initiated.

At first the wave propagated towards the wall, with imperceptible

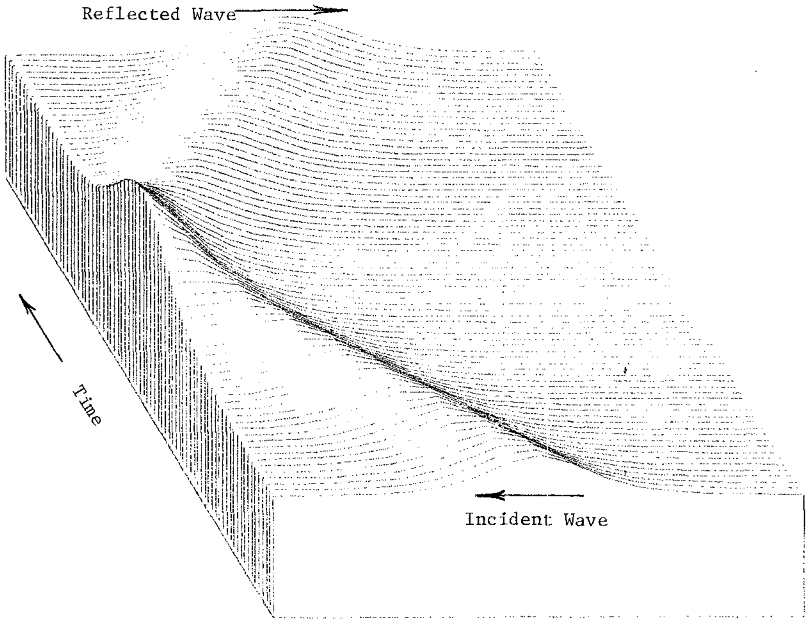

Figure 3. Surface elevation for a long wave of height 0.3 of the mean depth (approximating a solitary wave of height 0.325 above the trough), being reflected by a vertical wall at the left. Profiles are shown for successive times. Vertical exaggeration ≈ 4:1.

effect on the main body of the wave, as can be seen in Fig. 3, justifying the finite computational length taken. At the wall, the water level slowly rose until it was almost as high as the crest. After this stage, events happened very quickly, the crest "snapped through" to the wall and quickly reared up to more than twice the height of the incident wave. Far away from the wall, all the fluid was sensibly undisturbed. At its highest elevation, the crest at the wall became rather more curved, and for the highest waves became quite sharp. It is in this limit, at this stage, that the use of the Fourier method becomes questionable. The sharp crest of the breaking wave of limiting height cannot be described by the present method which depends on functions and derivatives having no discontinuities. Whereas a real wave may exhibit breaking at the crest for very high waves the numerical wave showed no such behaviour and can provide no breaking criterion. However, up until this point, the solutions were considered to be very accurate since total mass and energy of the fluid were conserved to at least four significant figures. In addition, when compared with experimental results and previous computational results for the maximum runup at the wall on Fig. 4, it can be seen that the present method gave results which agreed very closely, perhaps surprisingly in view of the fact that runup depends critically on whether the waves break or not.

The maximum force and moment exerted on the wall were calculated by obtaining the pressure on the wall at 21 equally-spaced points between the bottom and the crest and integrating using Simpson's rule. These quantities depend on the whole flow field and should be much less dependent on breaking at the crest than might be expected for the runup. Results from the numerical experiments are shown in Fig. 5, on which least-square parabolas have been fitted to each set of points, but with the condition that each pass through the zero amplitude hydrostatic results in which Force on wall = $\frac{1}{2}\rho g h^2$, where h is the undisturbed depth of water, and Moment about the toe of the wall = $\frac{1}{6}\rho g h^3$.

If H is the height of the incident wave crest above the undisturbed fluid, these results are:

Maximum Force on Wall F:

$$\frac{F}{\rho g h^2} = \frac{1}{2} + 2.25\left(\frac{H}{h}\right) - 0.42\left(\frac{H}{h}\right)^2$$

Maximum Moment on Wall about Toe:

$$\frac{M}{\rho g h^3} = \frac{1}{6} + 1.23\left(\frac{H}{h}\right) + 0.80\left(\frac{H}{h}\right)^2.$$

From the figure it can be seen that these empirical curves agree closely with all numerical results and should provide convenient criteria for design purposes.

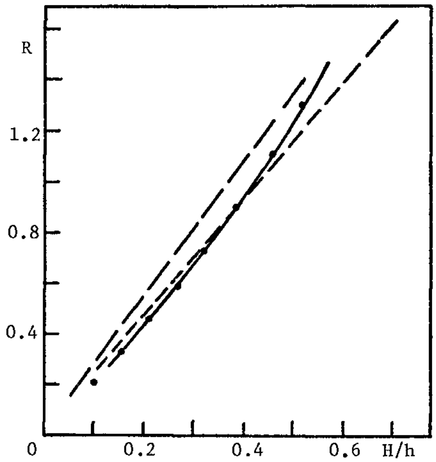

Figure 4. Maximum run-up at the wall R = (ηmax/h)-1, where h is undisturbed depth, plotted against incident wave height H/h. Points (•): numerical results from present method. (————): mean of experimental results reported by Chan & Street. (— — —): mean of experimental results from wave-wall reflection, and (—— ——): wave-wave interaction, both from Maxworthy (1976).

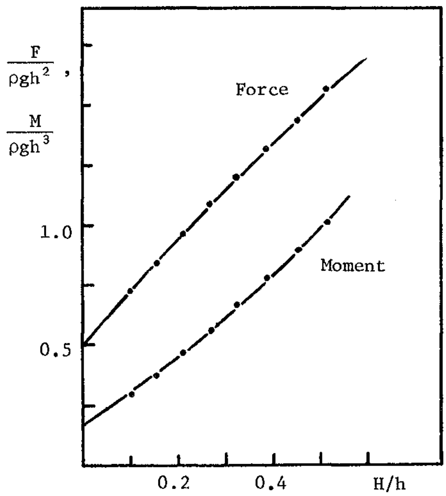

Figure 5. Maximum force and moment on the wall, plotted against incident wave height. The points (•) are numerical results from the present method to which curves have been fitted, as described in the text, shown by solid lines.

As the water flowed back down the wall, the process described above was qualitatively reversed, until the wave was totally reflected and was travelling in the opposite direction. Chan and Street (1970) reported that at this stage "the wave had exactly the same surface profile as its corresponding incident wave". However, in the present work, the reflected wave was not the same as the incident wave. This is not clear from the space-time plot on Fig. 3, as differences are relatively small. Fig. 6 shows rather more clearly the differences that were observed - the reflected wave has a depression behind it, it has a slightly steeper front face, it is lower, and in apparent contradiction of this last fact, it is travelling faster! These differences are not numerical errors: the accuracy of the method is shown by the fact that mass and energy of the water were conserved to within 10^{-4}, even for the highest waves reported. Rather, some differences between the waves is to be expected, for all the governing equations are highly nonlinear and it would be remarkable if two solitary waves of height 33% of the depth should collide, the combined crest grow to a height of 73%, and then each wave pass out through the other, without some nonlinear interaction changing the form of each. A number of details such as the time lag experienced in the interaction, the change in wave height and speed and other details of the nonlinear interaction are of little engineering importance and are being written up for publication elsewhere.

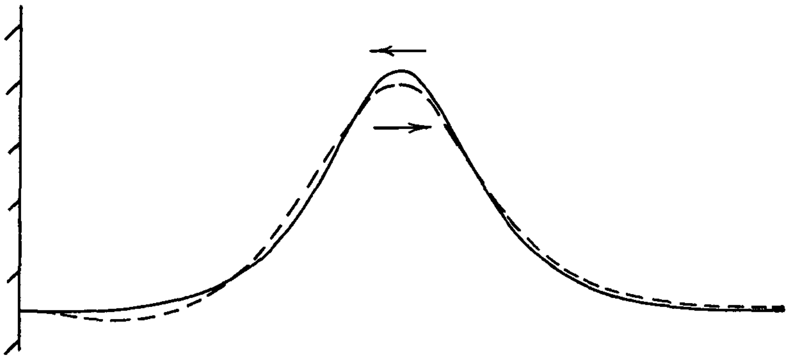

Figure 6. Comparison of surface profiles for an incident wave (————) of height 0.3 of the mean depth travelling towards the wall, and the wave after reflection (— — — —). Vertical exaggeration 16:1.

References

Byatt-Smith, J.C.B. 1971 An integral equation for unsteady surface waves and a comment on the Boussinesq equation, J. Fluid Mech. 49, 625-633.

Chan, R.K-C. & Street, R.L. 1970 A computer study of finite-amplitude water waves, J. Comp. Phys. 6, 68-94.

Cokelet, E.D. 1977 Steep gravity waves in water of arbitrary uniform depth, Phil. Trans. Roy. Soc. Lond. A 286, 183-230.

Dean, R.G. 1965 Stream function representation of nonlinear ocean waves, J. Ceophys. Res. 70, 4561-4572.

Fenton, J.D. 1979 A high-order cnoidal wave theory, J. Fluid Mech. 94, 129-161.

Le Méhauté, B., Divoky, D., & Lin, A. 1968 Shallow water waves : a comparison of theories and experiments, Proc. 11th Conf. Coastal Engng.

Longuet-Higgins, M.S. & Cokelet, E.D. 1976 The deformation of steep surface waves on water I. A numerical method of computation, Proc. R. Soc. Lond. A 350, 1-26.

Longuet-Higgins, M.S. & Fenton, J.D. 1974 On the mass, momentum, energy and circulation of a solitary wave II. Proc. R. Soc. Lond. A 340 471-493.

Maxworthy, T. 1976 Experiments on collisions between solitary waves, J. Fluid Mech. 76, 177-185.

Rienecker, M.M. & Fenton, J.D. 1980 A Fourier approximation method for steady water waves, J. Fluid Mech. (In Press).

Skjelbreia, L. & Hendrickson, J. 1961 Fifth order gravity wave theory, Proc. 7th Conf. Coastal Engng.