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On the Application of Steady Wave Theories

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SUMMARY Some recent developments and new results for the theories of steadily progressing waves and their application to marine problems are discussed. It is shown that most applications of Stokes and cnoidal theories have been incorrect at first order, and if neither the wave speed nor the current nor the mass flux is known, no theory higher than first order should be used. A new Stokes theory is presented, which is rather simpler to apply than previous theories. An existing cnoidal theory is modified to allow for the effects of current, and a number of recent simple formulae are presented for calculating the necessary elliptic functions and integrals. Finally, a recent Fourier approximation numerical method is discussed.

1 INTRODUCTION

The usual approximation made to solve the problem of waves propagating without rapid changes is that each wave is one of an infinitely long periodic wave train which propagates without change of shape, in water of constant depth. For seas in which there is a dominant wave period, such as swell, and where reflection and diffraction are not important, this should be quite a good approximation to the actual unsteady problem. Evidence suggests that individual waves appear to be rather stable, and the effects of nonlinear interactions are rather weak.

The problem of a steadily-progressing wave train has received considerable attention over the years. Of the various theories and numerical methods, there are three which stand out as being rational approximations, whose accuracy can be precisely quantified, which can be extended to higher orders, and whose applicability has been extensively tested. These are Stokes theory, cnoidal theory, and the numerical methods widely known as "stream-function" methods, but which are more appropriately known as Fourier approximation methods. Recent developments and new results in each of these areas will be described in this paper, showing how the application of each has become somewhat simpler in recent years. Before this is done, however, a feature will be discussed which has been commonly neglected by all wave theories, with the result that the application of most has been incorrect at first order.

2. DOPPLER SHIFTING OF THE WAVE PERIOD

2.1 Wave speed and period

Contrary to the implicit assumptions of many theories, no wave theory can predict the actual wave speed, for in most marine situations there is a finite current. The wave speed relative to an observer depends on the value of the current at that point; the waves travel faster with the current than against it. Thus, for waves of a given length, the apparent period measured by the observer (time interval between arrival of crests, say) depends on the current, or in familiar physics terms, the apparent period is Doppler-shifted. Without explicit allowance for the current it is not consistent to use the wave period in the calculation of a solution. Any solution thus obtained would be somewhat in error at first order, and it would be a waste of time to use any theory higher than that.

Most wave theories give formulae for either or both of two types of mean fluid speed in the water. Consider a frame of reference (X,Y), with X in the direction of propagation of the waves and Y vertically upwards, with the origin on the bed. The waves travel in the X direction at speed c relative to this frame. Consider also a frame of reference (x,y) moving with the waves at speed c, such that $X = x + ct$, $Y = y$, where t is time. In the (x,y) frame all fluid motion is steady, and consists of a flow in the negative x direction underneath the stationary wave profile. The mean fluid speed for a constant value of y over one wavelength λ is denoted by $\bar{u}$. If c_E is the measured time-mean horizontal fluid velocity at a point in the arbitrary (X,Y) frame through which the waves are passing (the Eulerian mean, hereinafter referred to as the current underneath the waves), then

$$c_E = c - \bar{u}. \tag{2.1}$$

Stokes' first definition of wave speed is the speed relative to a frame in which the current is zero, that is, in which $c_E = 0$, giving $c = \bar{u}$. In most presentations of wave theory it has been the quantity $\bar{u}$ which has been referred to as the "wave speed".

The second type of mean fluid speed is the depth-integrated mean velocity of the fluid under the waves. If Q is the volume flow rate per unit span underneath the waves in the (x,y) frame, the depth-averaged mean fluid velocity is $-Q/d$, where d is the mean depth. This velocity is negative because it is in the negative x direction. In the arbitrary (X,Y) frame, the depth-averaged mean fluid speed c_S, the "Stokes drift velocity" or the "mass-transport velocity" is given by

$$c_S = c - Q/d. \tag{2.2}$$

If there is no mass transport, $c_S = 0$, then Stokes' second definition of the wave speed, that relative to a frame in which there is no mass transport, gives $c = Q/d$.

In general, neither of Stokes' first or second definitions is necessarily the wave speed;

the waves can travel at any speed. In any particular marine situation, the overall physical problem will impose a certain value of c_E or c_S on the wave field, thus determining the wave speed.

2.2 The first step in applications

In most applications where a wave theory is to be used, the design parameters which are specified are the mean water depth d, the crest-to-trough wave height H, and the wave period τ. As described above, this is a Doppler-shifted period, and to apply any theory properly it is necessary to know in addition the wave speed c or the current c_E or the mass transport speed c_S.

Each wave theory gives formulae for $\bar{u}$ and for Q as functions of depth, wave height and of wavelength, which is usually unknown initially. Using (2.1) for example, and with the identity $\lambda = c\tau$, a nonlinear transcendental equation is obtained:

$$c_E - \frac{\lambda}{\tau} + \bar{u}(H,d,\lambda) = 0. \tag{2.3}$$

If H, d, τ and c_E are known, this nonlinear transcendental equation can be solved numerically for the wavelength, as the first step in the application of any theory. If, on the other hand, c_S is known, then an equation similar to (2.3) can be obtained and solved for the wavelength. Once λ is known, the rest of the theory can be applied. Clearly, if neither c_E nor c_S is known, then there is no way that either equation can be solved. As a sensible approximation, however, it would be reasonable to set $c_E = 0$, or $c_S = 0$, as has been implicitly done by most theories. If c_E (or c_S) is small compared with c, the computed value of λ would be accordingly accurate, however in this case there would be no justification in using any theory higher than first order. More importantly, however, the calculated unsteady water velocities, often used for design purposes, would have errors of the magnitude of the current. It is clearly important to know and to use the current or the mass transport velocity in applications.

3 STOKES′ THEORY

3.1 Introduction

Stokes assumed that all variation in the x direction can be represented by Fourier series, and that the coefficients in these series can be written as perturbation expansions in terms of a parameter which increases with wave height. Explicit fifth-order expressions for practical application have been given by De (1955), Chappelear (1961) and Skjelbreia & Hendrickson (1961). Each of these is for the special case $c_E = 0$, and so application of each has been inaccurate at first order if the current was not zero. In addition it has been shown by Fenton (1983) that the theories of De and Skjelbreia & Hendrickson contain errors at fifth order. These theories are presented in terms of length scales which are unknown initially, so the first step of any application requires the numerical solution of two or three simultaneous transcendental equations, which has been found to be difficult. Use of the actual wave height H and the actual depth d in the theory would necessitate the solution of a single equation only, as described in Section 2. This has been done by Tsuchiya & Yamaguchi (1972), who obtained a fourth-order solution for the special case $c_S = 0$.

Fenton (1983) has developed a fifth-order Stokes theory which requires solution of a single equation for the initial step, and which makes explicit allowance for the specification of a value of either c_E or c_S. It was shown that if neither is known then application of the theory is incorrect at first order, following the arguments of Rienecker & Fenton (1981).

Also shown, following Ursell, was that whereas the nominal expansion parameter in Stokes′ theory is proportional to H/λ, in shallow water the effective expansion parameter becomes $H\lambda^2/d^3$, so that for long waves (λ/d large) this parameter becomes large, and higher-order terms become large. Stokes′ theory should not be applied to waves which are longer than ten times the water depth which, nevertheless, is a fairly generous limitation. For waves longer than that, cnoidal theory should be used.

3.2 Presentation and application of theory

The following is a presentation of the theory as given by Fenton (1983). All expressions are given as functions of the dimensionless depth kd, and the dimensionless wave height kH/2, where $k = 2\pi/\lambda$ is the wavenumber. Each equation is given in Appendix A, and the formulae for the dimensionless coefficients used in those equations are given in Appendix B.

Equation (A.1) is that for the mean fluid speed along a line of constant elevation, $\bar{u}$. Using this, and $\lambda = c\tau$ gives (A.2), corresponding to (2.3), a nonlinear equation to be solved for the wavenumber k. If c_E is known, then (A.2) can be solved, or if c_S is known, the expression for Q (A.3) can be used to give equation (A.4), to be solved for k. An initial estimate for k can be obtained by considering the (identical) linearized versions of (A.2) and (A.4), for c_E or c_S small, where c_x means that either c_E or c_S can be substituted:

$$k \simeq \frac{4\pi^2}{\tau^2 g}\left(1 - \frac{4\pi c_x}{g\tau}\right).$$

From this initial estimate, the usual techniques for solving nonlinear equations can be used, such as trial and error, bisection, or the secant method. Then, the values of $\bar{u}$ from (A.1), Q from (A.3), and the wave speed c can be calculated from (A.5); the rest of the theory can then be used.

Expression (A.6) gives the velocity potential, from which the velocity components are obtained: $U = \partial\Phi/\partial X$ and $V = \partial\Phi/\partial Y$. The elevation of the free surface above the bed, $\eta(X,t)$ is given by (A.7), and the pressure at any point, $p(X,Y,t)$ is given by (A.8), where ρ is the fluid density, and where the value of the Bernoulli constant R is given by (A.9). In deep water the expressions (A.6–8) become (A.6*–8*) as shown, in which Y_* is the elevation above the mean water level, and the free surface is at $Y_* = \eta_*$.

In Fenton (1983), the fifth-order results are compared with experimental results and with high-order theory, and found to be accurate over a wide range of wave heights, provided the wave length is less than about ten times the water depth. Previous use of the incorrect condition $c_E = 0$ instead of $c_S = 0$ in comparison with closed wave tank experimental results was found to give errors of 20–40% for the fluid velocities.

4 CNOIDAL THEORY

4.1 Introduction

The cnoidal approximation to the steady water wave problem follows from a shallow water approximation, in which it is assumed that the waves are much longer than the water is deep. A first-order solution shows that the surface elevation is proportional to cn^2, where $cn(z|m)$ is a Jacobian elliptic function of argument z and modulus m and gives the name to the theory. This solution shows the long flat troughs and narrow crests, characteristic of waves in shallow water. In the limit as $m \to 1$, the solution correspons to the infinitely long solitary wave.

An explicit fifth-order theory has been given by Fenton (1979) which, like most of the Stokes theories described above, makes the implicit assumption that $c_E = 0$. In Section 4.2 it will be shown how Fenton's theory can be modified so as to be able to handle the more general cases where c_E or c_S may have a specified non-zero value.

It should be noted that cnoidal theory breaks down in deep water in a manner complementary to that of Stokes theory in shallow water. Fenton (1979) showed that whereas the nominal expansion parameter is H/d, the ratio of wave height to water depth, the effective parameter is H/md, where m is the parameter of the elliptic functions used. It can be simply showed that as deeper water is considered, this quantity varies like $(d/\lambda)^2$, so that the theory is clearly invalid for d/λ large.

4.2 Application of the theory

The theory described here is that of Fenton (1979), subsequently referred to as I. Most of the expressions given in I can be used directly and will not be reproduced here. The main purpose of this section is to extend the theory so that it can be applied to more general situations. This will be done here explicitly to second order.

As m is unknown initially, the first step in application is to solve for it, as outlined by equation (2.3), in a similar way to which (A.2 or 4) must be solved for the wavenumber k. In Appendix A of I, an expansion for the wavelength λ/d was given in terms of H/d, the parameter m, and the complete elliptic integrals of the first and second kind, K(m) and E(m) respectively. This expansion may be used directly, but to obtain the expansion for $u/(gd)^{1/2}$ it is necessary to use the results given in Tables A3 and B6 of I. Performing the necessary series manipulations and using the expression $\lambda = c\tau$ gives equation (C.1) shown in Appendix C here, where e(m) is the ratio E(m)/K(m). If the current c_E, depth d, wave period τ, and wave height H are known, this may be solved for the parameter m. As an initial estimate, it is possible to show that m is approximately given by

$$m \simeq 1 - 16 \exp[-(\frac{3gH\tau^2}{4d^2})^{1/2}],$$

for long waves, which shows how m approaches 1 for that case.

If c_S is known, then a similar expression may be developed from (2.2) and Table B3 of I, and is given as (C.2) in Appendix C. Having solved for m, the value of the trough depth h may be found from Table A3 of I, which to second order gives (C.3) here. The rest of the results given in Appendix B of I can now be directly used. Having found m correct to second order here, it would be rational only to use second order expressions from that Appendix of course.

4.3 Calculation of elliptic functions and integrals

For the waves where cnoidal theory is preferable to Stokes theory, m is very close to 1. Unfortunately in this limit the expressions for elliptic functions and elliptic integrals given in standard reference books are slowly convergent. The difficulty of calculating these quantities has been something of a disincentive to the use of cnoidal theory. Recently, however, Fenton & Gardiner-Garden (1982) have presented a number of formulae for elliptic functions and integrals which are most rapidly convergent in the long wave limit $m \to 1$. Some of these converge so rapidly that it is necessary to take only one term of the series.

Below are given formulae for the elliptic integrals K(m) and E(m), and the functions $cn(z|m)$, $sn(z|m)$, and $dn(z|m)$, which are used in the theory described above. The formulae given here are most accurate in the long wave limit $m \to 1$, but are generally accurate to five significant figures for $m > 1/2$. (For the case $m < 1/2$, when the cnoidal theory should be applied with great care, reference can be made to Fenton & Gardiner-Garden (1982) for the usual complementary formulae which work best when m is small.) Taking only the leading terms of various expressions gives the approximate result for K(m):

$$K(m) \simeq \frac{2}{(1 + m^{1/4})^2} \ln \frac{2(1 + m^{1/4})}{1 - m^{1/4}},$$

which is a remarkably simple result which seems not to have been given elsewhere. To a similar accuracy, the complementary elliptic integral of the first kind, $K'(m) = K(1-m)$ is given by

$$K'(m) \simeq \frac{2\pi}{(1 + m^{1/4})^2}.$$

The ratio $e(m)=E(m)/K(m)$ is given by

$$\frac{E(m)}{K(m)} \simeq \frac{2-m}{3} + \frac{\pi}{2KK'} + 2(\frac{\pi}{K'})^2[\frac{-1}{24} + \frac{q_1^2}{(1-q_1^2)^2}],$$

where $q_1(m)$ is the complementary nome $q_1 = \exp(-\pi K/K')$. Finally, the elliptic functions are given by the following:

$$sn(z|m) \simeq m^{-1/4} \frac{\sinh w - q_1^2 \sinh 3w}{\cosh w + q_1^2 \cosh 3w},$$

$$cn(z|m) \simeq \frac{1}{2}(\frac{m_1}{mq_1})^{1/4} \frac{1 - 2q_1 \cosh 2w}{\cosh w + q_1^2 \cosh 3w},$$

and

$$dn(z|m) \simeq \frac{1}{2}(\frac{m_1}{q_1})^{1/4} \frac{1 + 2q_1 \cosh 2w}{\cosh w + q_1^2 \cosh 3w},$$

in each of which $w = \pi z/2K'$.

5 FOURIER APPROXIMATION METHOD

A limitation to the use of both the Stokes and cnoidal theories is that they are not accurate for all waves. Neither of the fifth-order theories described above is accurate for high waves. To describe them properly it is necessary to take very high-order expansions and to use convergence-enhancement procedures, necessitating the use of extensive computation. A more fundamental limitation is that neither is accurate for all water depths: the Stokes theory is most accurate for waves in deeper water and breaks down in shallow water; while the cnoidal theory which can describe long waves is not applicable to deep water.

The basic form of the Stokes solution is a Fourier series, and so a reasonable step would be, instead of assuming perturbation expansions for the coefficients in the series, to find them numerically for a particular wave by solving the full nonlinear equations. This approach would be expected to break down in the limit of very long waves, when the Fourier spectrum of coefficients would be broad-banded and many terms would have to be taken. Also, as the highest waves are approached, the crest becomes more and more sharp, and the coefficients would also decrease rather more slowly. Despite these limitations, it would be expected to be very much more accurate than either of the perturbation approaches.

This is the essence of the methods of Chappelear (1961), Dean (1965) - known as the stream function method, and of Rienecker & Fenton (1981). The latter work contains a number of extra features which will be described and discussed here. (i) Finite values of c_E or c_S may be specified, and indeed must be specified if the solution is to have any significance, as described above. (ii) The only numerical approximation made is that the Fourier series is truncated. (iii) Solution of the resulting equations is by the rapidly-convergent Newton's method. (iv) The method shows better convergence properties, and can be applied to waves in deeper water, by including a factor cosh jkd in the j'th Fourier coefficient. (v) The method can describe the highest waves in deeper water, but for waves of finite height it is necessary to approach the solution by assuming an initial estimate obtained by extrapolation from converged solutions for lower waves. Otherwise the method does not converge. (vi) In shallow water it is still quite accurate for wavelengths of some tens of times the water depth, but loses accuracy as yet longer waves are considered. (vii) There are some small but important errors in the paper. Near the bottom of page 124, $\partial f_i/\partial B_o$ should be equal to $+\eta_m$, $\partial f_i/\partial B_o = +u_m$ near the top of page 125, and near the bottom of the same page, B_1 should be equal to $+H/2ck$.

In general, the method works well and gives accurate results. It is possibly the best way of solving the steady wave problem, but it is based on a numerical method which requires some computer programming, unlike the use of approximate theories which usually involve the relatively simple direct evaluation of quantities, as they are presented as an explicit solution. In problems where the waves are not very high or where great accuracy is not required, it is more reasonable to use an approximate explicit solution.

6 CONCLUSIONS

The three main approaches to the steady wave problem have been discussed, and some new results have been presented. It has been emphasized that insufficient recognition has been given to the fact that the wave speed (and hence the wave period) depends on the current on which the wave is travelling. Results presented include (i) A Stokes theory which uses the actual wave height in the expansion parameter, which makes practical application rather simpler. It is noted that some existing theories are wrong, (ii) An existing cnoidal theory is modified to allow for the more general case when the waves travel on a current. Some formulae are presented so that the elliptic functions and integrals may be more simply calculated for cnoidal waves, (iii) A recent Fourier approximation method has been briefly discussed.

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APPENDIX A: Equations for Stokes theory.

$$\bar{u}(k/g)^{1/2} = C_o + \left(\frac{kH}{2}\right)^2 C_2 + \left(\frac{kH}{2}\right)^4 C_4 + \ldots, \tag{A.1}$$

$$\left(\frac{k}{g}\right)^{1/2} c_E - \frac{2\pi}{\tau(gk)^{1/2}} + C_o(kd) + \left(\frac{kH}{2}\right)^2 C_2(kd) + \left(\frac{kH}{2}\right)^4 C_4(kd) + \ldots = 0, \tag{A.2}$$

$$Q(k^3/g)^{1/2} = C_o kd + \left(\frac{kH}{2}\right)^2 (C_2 kd + D_2) + \left(\frac{kH}{2}\right)^4 (C_4 kd + D_4) + \ldots, \tag{A.3}$$

$$\left(\frac{k}{g}\right)^{1/2} c_S - \frac{2\pi}{\tau(gk)^{1/2}} + C_o(kd) + \left(\frac{kH}{2}\right)^2 \left[C_2(kd) + \frac{D_2(kd)}{kd}\right] + \left(\frac{kH}{2}\right)^4 \left[C_4(kd) + \frac{D_4(kd)}{kd}\right] + \ldots = 0, \tag{A.4}$$

$$c = \bar{u} + c_E = \frac{Q}{d} + c_S , \tag{A.5}$$

$$\Phi(X,Y,t) = (c-\bar{u})X + C_o(g/k^3)^{1/2}\sum_{i=1}^{5}\left(\frac{kH}{2}\right)^i \sum_{j=1}^{i} A_{ij}\cosh jkY \sin jk(X-ct) + \ldots, \tag{A.6}$$

$$\begin{aligned} k\eta(X,t) = {} & kd + \left(\frac{kH}{2}\right)\cos k(X-ct) + \left(\frac{kH}{2}\right)^2 B_{22}\cos 2k(X-ct) \\ & + \left(\frac{kH}{2}\right)^3 B_{31}(\cos k(X-ct) - \cos 3k(X-ct)) + \left(\frac{kH}{2}\right)^4 (B_{42}\cos 2k(X-ct) + B_{44}\cos 4k(X-ct)) \\ & + \left(\frac{kH}{2}\right)^5 (-(B_{53}+B_{55})\cos k(X-ct) + B_{53}\cos 3k(X-ct) + B_{55}\cos 5k(X-ct)) + \ldots, \end{aligned} \tag{A.7}$$

$$p(X,Y,t)/\rho = R - gY - \frac{1}{2}[(U-c)^2 + V^2], \quad Rk/g = \frac{1}{2}C_o^2 + kd + \left(\frac{kH}{2}\right)^2 E_2 + \left(\frac{kH}{2}\right)^4 E_4 + \ldots, \tag{A.8, A.9}$$

$$\begin{aligned} (k^3/g)^{1/2}\Phi(X,Y_*,t) = {} & cX(k^3/g)^{1/2} - kX\left[1 + \frac{1}{2}\left(\frac{kH}{2}\right)^2 + \frac{1}{8}\left(\frac{kH}{2}\right)^4\right] + \left(\frac{kH}{2}\right)e^{kY_*}\sin k(X-ct) \\ & - \frac{1}{2}\left(\frac{kH}{2}\right)^3 e^{kY_*}\sin k(X-ct) + \frac{1}{2}\left(\frac{kH}{2}\right)^4 e^{2kY_*}\sin 2k(X-ct) \\ & + \left(\frac{kH}{2}\right)^5\left(\frac{-37}{24}e^{kY_*}\sin k(X-ct) + \frac{1}{12}e^{3kY_*}\sin 3k(X-ct)\right) + \ldots, \end{aligned} \tag{A.6*}$$

$$\begin{aligned} k\eta_*(X,t) = {} & \left(\frac{kH}{2}\right)\cos k(X-ct) + \frac{1}{2}\left(\frac{kH}{2}\right)^2\cos 2k(X-ct) + \frac{3}{8}\left(\frac{kH}{2}\right)^3(\cos 3k(X-ct) - \cos k(X-ct)) \\ & + \frac{1}{3}\left(\frac{kH}{2}\right)^4(\cos 2k(X-ct) + \cos 4k(X-ct)) \\ & + \frac{1}{384}\left(\frac{kH}{2}\right)^5(-422\cos k(X-ct) + 297\cos 3k(X-ct) + 125\cos 5k(X-ct)) + \ldots, \end{aligned} \tag{A.7*}$$

$$\frac{p}{\rho} = \frac{g}{k}\left(\frac{1}{2} - kY_* + \frac{1}{2}\left(\frac{kH}{2}\right)^2 + \frac{1}{4}\left(\frac{kH}{2}\right)^4\right) - \frac{1}{2}[(U-c)^2 + V^2]. \tag{A.8*}$$

APPENDIX B: Formulae for dimensionless coefficients in Stokes theory in terms of hyperbolic functions of kd, including S = sech 2kd.

$$A_{11} = 1/\sinh kd, \quad A_{22} = 3S^2/(2(1-S)^2), \quad A_{31} = (-4-20S+10S^2-13S^3)/(8\sinh kd\,(1-S)^3)$$
$$A_{33} = (-2S^2+11S^3)/(8\sinh kd\,(1-S)^3), \quad A_{42} = (12S-14S^2-264S^3-45S^4-13S^5)/(24(1-S)^5)$$
$$A_{44} = (10S^3-174S^4+291S^5+278S^6)/(48(3+2S)(1-S)^5)$$
$$A_{51} = (-1184+32S+13232S^2+21712S^3+20940S^4+12554S^5-500S^6-3341S^7-670S^8)/(64\sinh kd\,(3+2S)(4+S)(1-S)^6)$$
$$A_{53} = (4S+105S^2+198S^3-1376S^4-1302S^5-117S^6+58S^7)/(32\sinh kd\,(3+2S)(1-S)^6)$$
$$A_{55} = (-6S^3+272S^4-1552S^5+852S^6+2029S^7+430S^8)/(6\sinh kd\,(3+2S)(4+S)(1-S)^6)$$
$$B_{22} = \coth kd\,(1+2S)/(2(1-S)), \quad B_{31} = -3(1+3S+3S^2+2S^3)/(8(1-S)^3)$$
$$B_{42} = \coth kd\,(6-26S-182S^2-204S^3-25S^4+26S^5)/(6(3+2S)(1-S)^4)$$
$$B_{44} = \coth kd\,(24+92S+122S^2+66S^3+67S^4+34S^5)/(24(3+2S)(1-S)^4)$$
$$B_{53} = 9(132+17S-2216S^2-5897S^3-6292S^4-2687S^5+194S^6+467S^7+82S^8)/(128(3+2S)(4+S)(1-S)^6)$$
$$B_{55} = 5(300+1579S+3176S^2+2949S^3+1188S^4+675S^5+1326S^6+827S^7+130S^8)/(384(3+2S)(4+S)(1-S)^6)$$
$$C_o = (\tanh kd)^{1/2}, \quad C_2 = (\tanh kd)^{1/2}(2+7S^2)/(4(1-S)^2)$$
$$C_4 = (\tanh kd)^{1/2}(4+32S-116S^2-400S^3-71S^4+146S^5)/(32(1-S)^5)$$
$$D_2 = -(\coth kd)^{1/2}/2, \quad D_4 = (\coth kd)^{1/2}(4+2S+8S^2-5S^3)/(8(1-S)^3)$$
$$E_2 = \tanh kd\,(2+2S+5S^2)/(4(1-S)^2), \quad E_4 = \tanh kd\,(8+12S-152S^2-308S^3-42S^4+77S^5)/(32(1-S)^5)$$

APPENDIX C: Equations for first step of cnoidal theory.

$$\begin{aligned} \frac{c_E}{(gd)^{1/2}} + 1 + \left(\frac{H}{md}\right)\left(1 - \frac{3e}{2} - \frac{m}{2}\right) + \left(\frac{H}{md}\right)^2\left(-\frac{11}{15} + \frac{19e}{12} - \frac{5e^2}{8} + m\left(\frac{11}{15} - \frac{19e}{24}\right) - \frac{3m^2}{20}\right) & \\ - \left(\frac{md^2}{3gH\tau^2}\right)^{1/2} 4K\left(1 + \frac{H}{md}\left(\frac{5}{4} - \frac{3e}{2} - \frac{5m}{8}\right)\right) + \ldots = 0, & \end{aligned} \tag{C.1}$$

$$\begin{aligned} \frac{c_S}{(gd)^{1/2}} + 1 + \left(\frac{H}{md}\right)\left(1 - \frac{3e}{2} - \frac{m}{2}\right) + \left(\frac{H}{md}\right)^2\left(-\frac{2}{5} + \frac{e}{4} + \frac{3e^2}{8} + m\left(\frac{2}{5} - \frac{e}{8}\right) - \frac{3m^2}{20}\right) & \\ - \left(\frac{md^2}{3gH\tau^2}\right)^{1/2} 4K\left(1 + \frac{H}{md}\left(\frac{5}{4} - \frac{3e}{2} - \frac{5m}{8}\right)\right) + \ldots = 0, & \end{aligned} \tag{C.2}$$

$$\frac{h}{d} = 1 + \left(\frac{H}{md}\right)(1-m-e) + \left(\frac{H}{md}\right)^2\left(-\frac{1}{2} + \frac{m}{2} + \frac{e}{2} - \frac{em}{4}\right) + \ldots, \tag{C.3}$$