INTERNATIONAL ASSOCIATION FOR HYDRAULIC RESEARCH
21ST CONGRESS, AUGUST 1985, MELBOURNE, AUSTRALIA

# A FAMILY OF SCHEMES FOR COMPUTATIONAL HYDRAULICS

(Subject 2.a)

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SUMMARY A method is developed for generating numerical schemes for the solution of hydraulic equations, and is applied to the one-dimensional equations for unsteady open-channel flow. Examples included are for the water-depth and velocity formulation, and the stage and discharge formulation with off-stream storage. The schemes seem to offer a number of advantages over existing methods, and are simply implemented. They are explicit, unconditionally stable, of specified accuracy (exact for linear equations, and first-order in the basic nonlinear form, but higher order schemes can be developed). A novel feature is that the time-stepping schemes are independent of the method of interpolation along the channel, for which any method can be used, leading to the ability to treat irregular channels and the propagation of bores.

## 1. INTRODUCTION

The problem of the numerical solution of the one-dimensional equations for unsteady open channel flow has generated a massive literature during the last 20 years, reflecting the importance of the problem and, perhaps, the lack of any single outstanding method. Space permits only a cursory discussion here; reviews and critiques of the methods used have been given by Liggett & Cunge (1975), Zoppou (1979), and by Zoppou & O'Neill (1981).

For many years the method of characteristics has been used to demonstrate the nature of solutions to the equations, to obtain solutions in some simple cases, to infer the boundary conditions which must be supplied, and in more recent years to solve the equations numerically. While it has performed admirably in the first three areas, it has been less successful as a numerical tool; "... the method is fundamental and appealing to mathematicians ... however there remain difficulties which have prevented its wide use for unsteady flow in channels" (Liggett & Cunge). In an effort to overcome some of the disadvantages of characteristics, several grid-orientated characteristic methods have been developed, where the characteristic equations are solved at each stage to provide information at regularly-spaced values of space and time.

Turning to finite difference schemes where there is no attempt to build in the "travelling wave" nature of the solutions, and where the differential equations are approximated by finite difference expressions, there are two main subdivisions - explicit and implicit. "Explicit schemes are uneconomic, inflexible and inferior to other available numerical models" was the conclusion of Zoppou & O'Neill. If implicit schemes are used, then unconditional stability can be obtained and large time steps used. However, there seem to be a number of disadvantages also attached to implicit methods. They are very complicated, and also have a number of undesirable numerical properties.

After a reading of the extensive literature in the subject area one cannot fail to be impressed by the plethora of methods, the complexity of their formulae, and their limited abilities. It is the object of this work to develop a method for the numerical solution of all formulations of the one-dimensional equations for unsteady open-channel flow. The family of schemes which result are related to the grid-orientated characteristic methods, however they have a number of special features which leaves them with practically none of the disadvantages which conventional methods possess. The expressions given, after some lengthy derivation, are simple, explicit, unconditionally stable, of given accuracy (exact for linear equations, otherwise of first order only, but a second order family of schemes will be given in a later work), valid for sub- and supercritical flows, they describe the propagation and interaction of bores, and treat irregular channels simply. This latter feature derives from an unusual feature of the method, that the time-stepping formulae given are independent of the method of spatial interpolation along the channel. In obtaining the results presented here, exponential splines were found to be an accurate and reliable means of interpolation.

## 2. DEVELOPMENT OF METHOD FOR u-h FORMULATION

Consider the equations governing the propagation of long disturbances, where the depth of water $h(x,t)$ and the mean fluid velocity $u(x,t)$ over the channel are the dependent variables and $x$ and $t$ are respectively the distance along the channel and time. The mass conservation equation is

$$\frac{\partial h}{\partial t} = -u\frac{\partial h}{\partial x} - \frac{A}{B}\frac{\partial u}{\partial x} + \frac{q}{B} - \frac{u}{B}A_x \qquad (2.1)$$

and the momentum equation is

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x} - u\frac{\partial u}{\partial x} + gS_0 - gS_f + \frac{q}{A}(u_i - u), \qquad (2.2)$$

where $A(x,h(x,t))$ is the area of cross-section of the channel, $A_x = \partial A/\partial x|_h$ is the rate of change of A with x due to changes in the channel section only, $B(x,h(x,t))$ is the width of the free surface, $g$ is gravitational acceleration, $S_0(x)$ is the slope of the channel bottom, decreasing elevation with x being taken as positive slope, $S_f(x,t)$ is the friction slope, always positive, for which empirical formulae are usually given, $q(x,t)$ is the volume inflow rate per unit length of channel, and $u_i(x,t)$ is the x-component of the velocity with which that inflow enters the channel.

The equations can be written in matrix form

$$\frac{\partial \mathbf{u}}{\partial t} = -F\,\frac{\partial \mathbf{u}}{\partial x} + \mathbf{q}, \qquad (2.3)$$

where $\mathbf{u}(x,t)$ is the vector of dependent variables, $F(x,t)$ a matrix, and $\mathbf{q}$ the vector containing the effects of geometric changes, inflow, bottom slope and friction:

$$\mathbf{u} = \begin{bmatrix} h \\ u \end{bmatrix}, \quad F = \begin{bmatrix} u & A/B \\ g & u \end{bmatrix},$$

and $\qquad (2.4)$

$$\mathbf{q} = \begin{bmatrix} q/B - uA_x/B \\ g(S_0 - S_f) + q(u_1 - u)/A \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix},$$

where $q_1$ and $q_2$ are introduced here for the terms shown.

The usual problem in computational hydraulics is, if $\mathbf{u}(x,t)$ is known, to find the solution at a later time $\mathbf{u}(x,t+\Delta)$, where $\Delta$ is a finite time step. In the absence of boundary conditions, an exact solution can be written as the infinite Taylor series

$$\mathbf{u}(x,t+\Delta) = \mathbf{u}(x,t)+\Delta\frac{\partial \mathbf{u}}{\partial t}(x,t)+\frac{\Delta^2}{2!}\,\frac{\partial^2\mathbf{u}}{\partial t^2}(x,t)+\ldots \qquad (2.5)$$

The system of governing partial differential equations (2.3) can be substituted for $\mathbf{u}(x,t)$. For time derivatives of higher degree, from (2.3) the approximation is made that, for $n \geq 2$,

$$\frac{\partial^n \mathbf{u}}{\partial t^n} \simeq (-F)^n\,\frac{\partial^n \mathbf{u}}{\partial x^n}. \qquad (2.6)$$

Substituting these results into the Taylor series (2.5) gives

$$\mathbf{u}(x,t+\Delta) = \left( I - \Delta F\,\frac{\partial}{\partial x} + \frac{\Delta^2}{2!}(-F)^2\frac{\partial^2}{\partial x^2} + \ldots\right) \mathbf{u}(x,t) + \Delta\mathbf{q}(x,t) + O(\Delta^2), \qquad (2.7)$$

where $I$ is the $2\times2$ identity matrix, and where the Landau order symbol $O(\Delta^2)$ has been used to show that neglected terms are of order $\Delta^2$, due to the approximation of (2.6). Equation (2.7) includes a differential operator of infinite order. While it might be thought strange to include an infinity of terms while neglecting terms of order $\Delta^2$, it has been shown by Fenton (1982,1983) that this scheme solves an important class of problems exactly, and has numerical properties which are very much nicer than conventional finite difference methods.

To implement the scheme (2.7) it would be necessary to perform an infinite series of matrix multiplications. This can be obviated if the matrix $F$ is diagonalized by writing it as $F = CDC^{-1}$, where $D$ is a diagonal matrix with elements $\lambda_1$ and $\lambda_2$ which are the eigenvalues of $F$, and $C$ is a matrix whose columns are the corresponding eigenvectors of $F$. Then, it can be shown that $F^n = CD^nC^{-1}$, and (2.7) becomes

$$\mathbf{u}(x,t+\Delta) = \left(I-\Delta CDC^{-1}\frac{\partial}{\partial x}+\frac{\Delta^2}{2!}CD^2C^{-1}\frac{\partial^2}{\partial x^2} + \ldots\right)\mathbf{u}(x,t) + \Delta\mathbf{q}(x,t) + O(\Delta^2), \qquad (2.8)$$

With $F$ as defined in (2.4), the eigenvalues are easily shown to be

$$\lambda_1 = u + \sqrt{gA/B}, \quad \lambda_2 = u - \sqrt{gA/B}. \qquad (2.9)$$

These terms now have an important physical significance, for the speed of long waves is given by $\sqrt{gA/B}$. Using the customary symbol $c$ for this quantity, the eigenvalues are

$$\lambda_1 = u + c, \quad \text{and} \quad \lambda_2 = u - c,$$

the local speeds at which disturbances travel on the flow, relative to a stationary observer.

With these eigenvalues, it is easily shown that

$$C = \frac{1}{2gc}\begin{bmatrix} c & c \\ g & -g \end{bmatrix}, \quad \text{with} \quad C^{-1} = \begin{bmatrix} g & c \\ g & -c \end{bmatrix}. \qquad (2.10)$$

Substituting this expression for $C^{-1}$ into (2.8), and using the fact that the only elements of $D^n$ are simply $\lambda_1^n$ and $\lambda_2^n$, on the diagonal,

$$\mathbf{u}(x,t+\Delta) = C\left(\begin{bmatrix} g & c \\ g & -c \end{bmatrix} - \Delta\begin{bmatrix} g\lambda_1 & c\lambda_1 \\ g\lambda_2 & -c\lambda_2 \end{bmatrix}\frac{\partial}{\partial x} + \frac{\Delta^2}{2!}\begin{bmatrix} g\lambda_1^2 & c\lambda_1^2 \\ g\lambda_2^2 & -c\lambda_2^2 \end{bmatrix}\frac{\partial^2}{\partial x^2} + \ldots\right)\mathbf{u}(x,t) + \Delta\mathbf{q}(x,t) + O(\Delta^2).$$

Grouping the matrices, this can be written

$$\mathbf{u}(x,t+\Delta) = C\begin{bmatrix} g\,\exp(-\Delta\lambda_1\partial/\partial x) & c\,\exp(-\Delta\lambda_1\partial/\partial x) \\ g\,\exp(-\Delta\lambda_2\partial/\partial x) & g\,\exp(-\Delta\lambda_2\partial/\partial x) \end{bmatrix} + \Delta\mathbf{q}(x,t) + O(\Delta^2),$$

in which the exponential expressions are used to represent their power series expansions

$$\exp(-\Delta\lambda_1\partial/\partial x) = 1-\Delta\lambda_1\partial/\partial x+\frac{1}{2!}\Delta^2\lambda_1^2\partial^2/\partial x^2-\ldots,$$

These infinite series of differential operators are the operators of Taylor series, and can be simply interpreted as Shift Operators $E(-\Delta\lambda_1)$ and $E(-\Delta\lambda_2)$, such that if such a shift operator acts on some function $f(x)$, the value of $f$ at a shifted value of $x$ is obtained: $E(-\Delta\lambda_1)\,f(x) = f(x - \Delta\lambda_1)$, etc.

Substituting these results into (2.8) gives

$$\mathbf{u}(x,t+\Delta) = C\begin{bmatrix} gE_+ & cE_+ \\ gE_- & -cE_- \end{bmatrix}\mathbf{u}(x,t) + \Delta\mathbf{q}(x,t) + O(\Delta^2), \qquad (2.11)$$

where $E_+$ and $E_-$ are defined by

$$E_+ = E(-\Delta\lambda_1) = E(-\Delta(u+c)),$$
$$\text{and} \quad E_- = E(-\Delta\lambda_2) = E(-\Delta(u-c)).$$

The shift operators now have an important and unexpected physical significance: to calculate the solution at $x$ at time $t+\Delta$, one uses information from time level $t$ at $x_+ = x - \Delta(u+c)$ and $x_- = x - \Delta(u-c)$, the points from which disturbances would originate, travelling at velocities $u(x,t)+c(x,t)$ and $u(x,t)-c(x,t)$ respectively, such that they would arrive at the point x at time $t+\Delta$. Although no mention has been made of characteristics in the present derivation, it seems that it implicitly acknowledges the characteristic-based form of the solution. After substitution for C into (2.11), and multiplying the matrices, the resulting individual component equations become

$$h(x,t+\Delta) = \frac{1}{2}(h_+ + h_-) + \frac{c(x,t)}{2g}(u_+ - u_-) + \Delta q_1(x,t) + O(\Delta^2),$$

and (2.12)

$$u(x,t+\Delta) = \frac{1}{2}(u_+ + u_-) + \frac{g}{2c(x,t)}(h_+ - h_-) + \Delta q_2(x,t) + O(\Delta^2),$$

in which $h_\pm = h(x_\pm, t)$, and $u_\pm = u(x_\pm, t)$, where $x_\pm = x - \Delta(u(x,t) \pm c(x,t))$.

Equations (2.12) are the fundamental scheme of this work. Function values at $(x,t+\Delta)$ are obtained from values at $(x_\pm,t)$, values which must be interpolated from known point values at time level t. As such, the equations provide an unusual method of solution. There is no attempt to approximate derivatives, they form an interpolation-only scheme for advancing the solution in time, while their approximation of the governing equations is implicit rather than explicit. The actual process of interpolation can be carried out by any means, and in this formulation there is total freedom so to do. There has been no low-order approximation using finite differences. The approximation is entirely in the time-stepping.

The method is most closely related to grid-orientated characteristic schemes. If straight line approximations to characteristics were to be used, they would yield schemes such as this. The important difference is, however, that the present approach can be used for systems where no convenient characteristic formulation exists. Such a case will be explored in Section 4.

## 3. INCLUSION OF BOUNDARY CONDITIONS

The basic scheme (2.12) makes no provision for the specification of boundary conditions. In the vicinity of a boundary, in general either $x_+$ or $x_-$ (or possibly both at an upstream supercritical boundary) will fall outside the domain of the reach of river or canal, and the above interpolation scheme can no longer be used. Here, the method is modified so that it automatically classifies the type of boundary and incorporates all of the necessary boundary conditions there.

If (2.11) is pre-multiplied by $C^{-1}$, the following two equations are obtained (choosing + or − in each case):

$$\begin{aligned} gh(x,t+\Delta) &\pm c(x,t)\, u(x,t+\Delta) \\ &= gh_\pm \pm c(x,t)\, u_\pm \\ &+ \Delta(gq_1 \pm cq_2)_{(x,t)} + O(\Delta^2) . \end{aligned} \qquad (3.1)$$

The presence of common coefficients (g and c(x,t)) in each of the lines of this equation further suggests the underlying characteristic nature of the present method, although not in the precise sense of the word: here the quantities $gh(X,T) \pm c(x,t)u(X,T)$ are constant (to first order in $\Delta$, except for modifications due to the forcing terms $q_1$ and $q_2$) on straight lines corresponding to the sets of points $(X,T)$ joining $(x,t+\Delta)$ and $(x_\pm,t)$, with gradients $dX/dT = u(x,t) \pm c(x,t)$. Here, these lines will be termed quasi-characteristics, and the two combinations of h and u termed quasi-invariants.

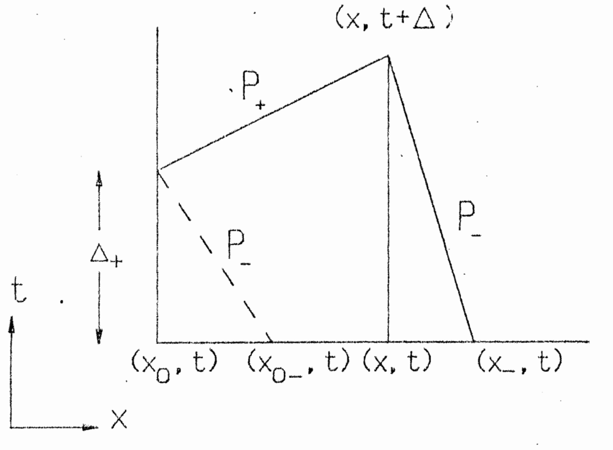


Figure 1: Use of quasi-characteristic formulation near a boundary.

Figure 1 shows a typical boundary encounter (for subcritical flow near the left end of a reach, $x = x_0$), where the quasi-characteristics corresponding to $x_\pm$ are labelled $P_\pm$ respectively. The point $(x_-,t)$ occurs within the reach, however $(x_+,t)$ is outside, and $P_+$ intersects $x = x_0$ at $t + \Delta_+$. A general formula can be given, that if a quasi-characteristic crosses a boundary $x_b$ ($= x_0$, or $x_n$ for the right hand boundary), then

$$\Delta_\pm = \Delta - \frac{x - x_b}{(u \pm c)_{(x,t)}} . \qquad (3.2)$$

It is possible to show that (3.1) can be rewritten for $(x_b, t+\Delta_\pm)$ instead of $(x_\pm,t)$ to give

$$\begin{aligned} gh(x,t+\Delta) &\pm c(x,t)\, u(x,t+\Delta) \\ &= gh(x_b,t+\Delta_\pm) \pm c(x,t)\, u(x_b,t+\Delta_\pm) \\ &+ (\Delta - \Delta_\pm)(gq_1 \pm cq_2)_{(x,t)} + O(\Delta^2) . \end{aligned} \qquad (3.3)$$

This expression is valid for both left and right boundaries and for $x_+$ and/or $x_-$ lying outside those boundaries. If both are outside a particular boundary, that corresponds to inwards-directed supercritical flow ($|u| > |c|$), and values of h and u must be provided at the boundary $x_b$ for all values of t, either in the form of a supplied function or interpolating between supplied point values. Then, the right side of (3.3) can be evaluated for both + and −, giving two equations which can be solved for both h and u at $(x,t+\Delta)$.

On the other hand, if one of $x_+$ or $x_-$ (that one denoted by $x_s$) is outside the boundary, then the local flow is subcritical, and only one of h or u at $(x_b,t+\Delta_s)$ is to be specified. The other is obtained from (3.4), which is (3.1) written for the other quasi-characteristic emanating from $(x_b,t+\Delta_s)$, $P_{-s}$, shown dotted in Figure 1:

$$\begin{aligned} gh(x_b,t+\Delta_s) &- s\, c(x_b,t)\, u(x_b,t+\Delta_s) \\ &= gh(x_{b,-s},t) - s\, c(x_b,t)\, u(x_{b,-s},t) \\ &+ \Delta(gq_1 - s\, cq_2)_{(x_b,t)} + O(\Delta^2) , \end{aligned} \qquad (3.4)$$

quantities on the right side being obtained by interpolation at the same time level t (at $(x_{0-},t)$ on Figure 1). Now (3.3) can be used, with (3.1) for the other $P_{-s}$ characteristic to give two equations to be solved for h and u at $(x,t+\Delta)$.

The somewhat fussy nature of this boundary treatment is reminiscent of characteristic and explicit finite difference schemes, but in the present formulation it can be programmed rather concisely. In fact, all computational points, whether on or near the boundary, or interior points (when both equations of (3.1) are used), can be treated by the methods of this section, replacing the scheme (2.12).

## 4. METHOD FOR Q-z FORMULATION WITH OFF-STREAM STORAGE

Instead of the velocity u and local depth h as dependent variables, it is often more convenient to work with the discharge Q and the stage z. The equations are as given by Liggett & Cunge (1975) and slightly more generally by Zoppou (1979). Liggett (1968) has considered the effects of water overflowing the main banks, so that off-stream storage is provided, in which there is no flow in the streamwise direction. This provides some problems for the method of characteristics, in that invariants do not exist. However, in applications where there are effects due to slope, friction and inflow, they are not real invariants anyway, and the existence of off-stream storage does not seem to be such a problem. Here, the method developed above is modified to allow for off-stream storage. It will be seen that the essence of the method is unchanged.

As presented by Liggett, the only modification to the equations is the replacement of B by $B_s$ in the storage term. The mass conservation equation becomes

$$\frac{\partial z}{\partial t} = -\frac{1}{B_s}\frac{\partial Q}{\partial x} + \frac{q}{B_s}. \tag{4.1}$$

Taking the momentum equation (2.2), substituting Q/A for u, and where necessary substituting (4.1) gives

$$\frac{\partial Q}{\partial t} = -\left(gA - \frac{Q^2B}{A^2}\right)\frac{\partial z}{\partial x} - \frac{Q}{A}\left(1 + \frac{B}{B_s}\right)\frac{\partial Q}{\partial x} + \frac{Q^2}{A^2}A_x - gAS_f + q\left(u_i + \frac{Q}{A}\left(\frac{B}{B_s} - 1\right)\right). \tag{4.2}$$

The notation of section 2 can be used. Let

$$\mathbf{u}=\begin{bmatrix} z \\ Q \end{bmatrix}, \text{ and } \mathbf{q}=\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}=\begin{bmatrix} q/B_s \\ \frac{Q^2}{A^2}A_x - gAS_f + q\left(u_i + \frac{Q}{A}\left(\frac{B}{B_s} - 1\right)\right) \end{bmatrix},$$

then the coefficient matrix F is

$$F = \begin{bmatrix} 0 & 1/B_s \\ gA - Q^2B/A^2 & (1 + B/B_s)Q/A \end{bmatrix}, \tag{4.3}$$

In this case, the eigenvalues of F are

$$\lambda = \frac{1}{2}\left(1 + \frac{B}{B_s}\right)\frac{Q}{A} \pm \sqrt{\frac{1}{4}\frac{Q^2}{A^2}\left(1 - \frac{B}{B_s}\right)^2 + \frac{gA}{B_s}}. \tag{4.4}$$

In Section 2, when there was no off-stream storage, $B = B_s$, this gave $\lambda = u \pm \sqrt{gA/B}$, which was defined to be $\lambda = u \pm c$. Those eigenvalues had the important physical significance that they were the velocities with which disturbances propagated relative to a stationary observer. All such disturbances were carried along at a velocity u with the flow, and if they propagated downstream or upstream with speed c relative to the flow, then the velocities were u+c and u-c respectively.

In the present case where there is off-stream storage, such that $B \neq B_s$, it is reasonable to infer that the same physical significance attaches to the components of equation (4.4). Thus, it seems that the effect of the flow and off-stream storage is that disturbances are swept along with a velocity due to the stream flow, designated here by U, such that

$$U = \frac{1}{2}\left(1 + \frac{B}{B_s}\right)\frac{Q}{A}, \tag{4.5}$$

and the velocities of disturbances relative to this are $\pm c$, where c is given by

$$c^2 = \frac{1}{4}\frac{Q^2}{A^2}\left(1 - \frac{B}{B_s}\right)^2 + \frac{gA}{B_s}. \tag{4.6}$$

Thus (4.4) becomes simply

$$\lambda = U \pm c,$$

and the convention is adopted

$$\lambda_1 = U + c, \quad \text{and} \quad \lambda_2 = U - c.$$

In view of the above discussion, there seems to be nothing sacrosanct about reserving the symbol c for the quantity $\sqrt{gA/B}$ as has been done in previous works; from (4.4) it is more reasonable to use c as defined in (4.6).

After some manipulation, the matrices C and $C^{-1}$ can be obtained, and equation (2.11) used to give the time-stepping scheme:

$$\begin{aligned} z(x,t+\Delta) = {} & \tfrac{1}{2}\left[\left(1 - \tfrac{U}{c}\right)_{(x,t)} z_+ + \left(1 + \tfrac{U}{c}\right)_{(x,t)} z_-\right] \\ & + \left(2c(x,t)\,B_s(x,t)\right)^{-1}\left[Q_+ - Q_-\right] \\ & + \Delta q_1 + O(\Delta^2), \end{aligned}$$

and

$$\begin{aligned} Q(x,t+\Delta) = {} & \tfrac{1}{2}\left[\left(1 + \tfrac{U}{c}\right)_{(x,t)} Q_+ + \left(1 - \tfrac{U}{c}\right)_{(x,t)} Q_-\right] \\ & + \left(B_s(c^2 - U^2)/2c\right)_{(x,t)}\left[z_+ - z_-\right] \\ & + \Delta q_2 + O(\Delta^2), \end{aligned} \tag{4.7}$$

where subscripts (x,t) show that the whole term is evaluated at that point.

It is clear this scheme is, except for different coefficients of the quantities to be interpolated, very much the same as that given in equations (2.12) for the u-h formulation. The method presented in each of these two sections 2 and 4, seems to be of wide generality.

## 5. RESULTS

Some results are presented here for the case of a rectangular canal with a vertical wall at the right end, and a prescribed velocity as a function of time at the left. Forty computational points were used. Results for all cases show the water surface elevation at any instant by a single line, those above this corresponding to later times.

Figures 2 and 3 are for the weakly non-linear case, where a single ("cosine bell") wave is input, after which the velocity on the left boundary remains zero. In this case the wave is small (1/20 of the water depth) such that the situation is almost linear, and one would expect to see the wave being reflected back and forth with little change. In all figures the scheme (2.12) was used; in Figure 2 interpolation between grid points was straight-line. It can be seen that the method describes the propagation and reflection of the waves reasonably well, but with a rather large amount of numerical damping and diffusion of the wave (it has become lower and wider), due to the low accuracy of linear approximation (see Fenton

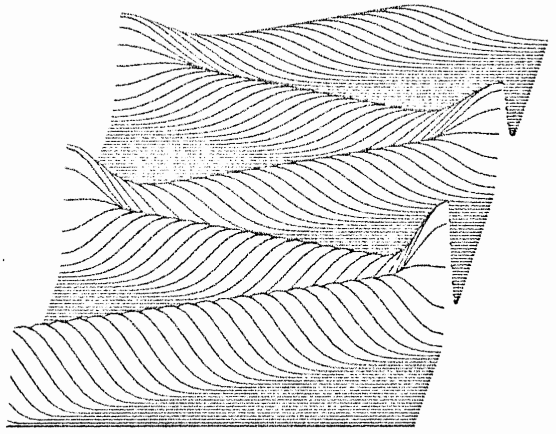

Figure 2: Propagation and reflection of a single wave, using linear interpolation.

(1982) for a physical explanation). It is, however, this level of approximation which is used widely throughout computational hydraulics; this might be the cause for some concern.

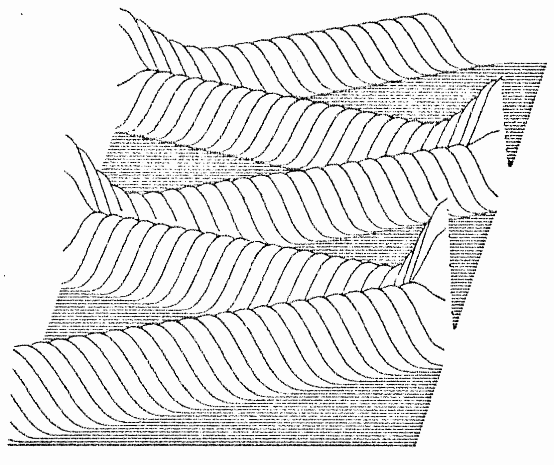

Figure 3: Propagation and reflection of a single wave, using exponential spline interpolation, in the near-cubic spline limit.

Figure 3 shows the same problem, but where exponential spline approximation (Tornow, 1982) has been used, with a relatively high level of approximation. There is some numerical damping (in this case and the previous one, this is primarily due to the rapid variation due to the reflection at the walls), however the performance of the scheme is substantially better than with linear interpolation.

The damping associated with linear interpolation can be an advantage where bores occur, as shown in Figure 4, where the inflow velocity at the left is initially the same as above, but having reached the maximum value, it remains constant. The amplitude now is half the depth, and as expected nonlinear steepening produces a bore (which, ideally, should be a vertical line). This

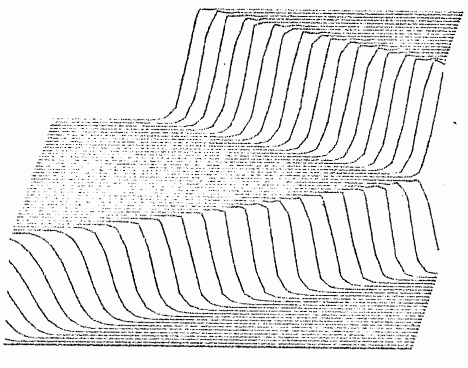

Figure 4: Propagation and reflection of a bore, using linear interpolation.

is then reflected from the right wall. The linear interpolation describes this situation quite satisfactorily, as did exponential spline interpolation in the relatively stiff approximation limit, although not shown here. In the cubic spline limit, with the ability of the interpolatory function to oscillate between data points, finite oscillations developed in the region of the bore. However, in most applications, where variation is relatively smooth, cubic splines would be preferred. Throughout computations for this paper, the author found exponential splines to be particularly useful, with their ability to become cubic splines in one parameter limit, and essentially piecewise-linear approximation in the other limit. They may have an important future in computational hydraulics.

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