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A simplified approach to reservoir routing

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SUMMARY A re-examination of procedures for reservoir routing is suggested. Use of the traditional implicit method for reservoir routing has obscured the fundamental simplicity of the problem, and introduces unnecessary complications and sources of inaccuracy. If it is recognised that reservoir routing involves solving a differential equation, then procedures can be made simpler and more flexible. Various numerical methods are examined and compared, and it is seen that almost any method gives acceptable accuracy. An alternative form of the governing equation in terms of the reservoir surface elevation is shown to have some advantages over the usual form involving storage volume. The presentation incorporates the case where outflow from the reservoir may be varied by control of valves or spillway gates. It is concluded that explicit methods are to be preferred to the traditional implicit method, and it is suggested that further implementations of the latter be discontinued.

1. INTRODUCTION

The storage equation governing rate of change of reservoir storage volume is

$$\frac{dS}{dt} = I(t) - Q(t, S), \tag{1}$$

in which S is the volume of water stored in a reservoir, t is time, I is the volume rate of inflow, which is a known function of time or known at points in time, and Q is the volume rate of outflow. To solve the equation it is necessary to relate the outflow to the storage volume, usually *via* the dependence of each on the elevation of the water surface, such that $Q(t, S)$ expresses the dependence of Q on S and t. In many discussions of the storage equation the outflow Q has been considered to depend only on S. However, where reservoir outflow is controlled by a valve or a spillway gate the outflow characteristics can be varied by human or automatic control. The outflow is then some known function of time t as well as S.

Equation (1) is a first-order differential equation for S as a function of t. It can be solved numerically by any one of a number of methods of varying complexity and accuracy, of which the most elementary are very simple. The fact that the problem is merely one of solving a differential equation, and the ease with which this can be done, seems not to have been exploited. The traditional method of solving equation (1), described in almost all books on hydrology, is relatively complicated. The differential equation is approximated by

$$I(t) + I(t+\Delta) + \frac{2S(t)}{\Delta} - Q(S(t)) \approx \frac{2S(t+\Delta)}{\Delta} + Q(S(t+\Delta)), \tag{2}$$

where Δ is a finite step in time, and where it has been assumed that there are no externally-controlled discharges, such that Q can be expressed as a function of S only. At a particular time t all the quantities on the left side can be evaluated. The equation is then a nonlinear equation for the single unknown quantity $S(t+\Delta)$, the storage volume at the next time step, which appears transcendentally on the right side. There are several methods for solving such equations and the solution is in principle not particularly difficult. However, textbooks at an introductory level are forced to present procedures for solving such equations (by graphical methods or by inverse interpolation) which tend to obscure with mathematical and numerical detail the underlying simplicity of reservoir routing. At an advanced level a number of practical difficulties may arise (Laurenson, 1986), such that in the solution of the nonlinear equation considerable attention may have to be given to pathological cases. As the methods are iterative, several function evaluations of the right side of equation (2) are necessary at each time step. Below it is shown that this method is one of a family of *implicit* methods of solution.

It is the aim of this paper to show that a formulation of the governing differential equation in terms of reservoir surface elevation has some advantages over equation (1) and that use of the traditional method (equation (2)) for reservoir routing has made solution rather more difficult than it need be. If it is recognised that reservoir routing is the numerical solution of a differential equation, then any one of a variety of methods can be used, which are simpler, more flexible, and require less computation. It is suggested that the traditional implicit method be not further implemented.

2. ALTERNATIVE STORAGE EQUATION

In addition to the S formulation already described, other forms of the differential equation can be simply obtained. If the reservoir surface elevation h changes by an amount dh, in the limit $dh \to 0$ the change in storage dS is given by

$$dS = A(h)\, dh, \tag{3}$$

where $A(h)$ is the plan area of the water surface at elevation h. Substituting into (1) and writing the outflow Q as a function of h (usually a simple power law or combinations of such laws), and as a function of t in the case of controlled discharges, an equivalent form of the storage equation is obtained:

$$\frac{dh}{dt} = \frac{I(t) - Q(t,h)}{A(h)}, \qquad (4)$$

which is a differential equation for the surface elevation itself. This equation has been presented by Chow *et al.* (1988, Section 8.3), and by Roberson *et al.* (1988, Section 10.7), but as a supplementary form to (1). In fact it has some advantages over that form, and this h formulation might generally be preferred. It makes no use of the storage volume S, which then does not have to be calculated.

Otherwise, if the traditional form (1) is used, then S has to be obtained as a function of h from the integral

$$S(h) = \int^{h} A(y)\,dy, \qquad (5)$$

by a low-order numerical approximation for various surface elevations. Then it is necessary to eliminate h between $Q(t,h)$ and $S(h)$, so that S can be expressed as a function of Q, usually by creating a table of pairs of corresponding values of Q and S. In those cases where the discharge is controlled, this has to be repeated every time a gate or valve is adjusted.

An apparent advantage of the h formulation is the avoidance of these steps and of the low-order numerical evaluation of the integral in (5). Usually where outflow is *via* outlet pipes and spillways, the discharge Q can be expressed as a simple mathematical function of h, usually involving terms like $(h - y_{outlet})^{1/2}$ and/or $(h - y_{crest})^{3/2}$, where y_{outlet} is the elevation of the pipe or tailrace outlet to atmosphere and y_{crest} is the elevation of the spillway crest. The dependence on t can be obtained by specifying the vertical gate opening or valve characteristic as a function of time, usually as a coefficient multiplying these powers of h.

In the same way that the dependence of Q on S is usually represented by a table of pairs of values, in general the h formulation requires a similar table for A and h, to give $A(h)$, obtained from planimetric information from contour maps. However, this process is less liable to error for the reasons set out in the following argument: almost every spillway has a crest level considerably above the bottom of the reservoir, and the total operating height range of the spillway is small compared with the total depth of the reservoir. Whereas there might be a number of contour intervals used to calculate the volumes in the reservoir, there might be relatively few in the height range of the spillway. If the conventional S formulation is used, then the outflow Q has to be calculated as a function of S in that range, and there might be few values of S for that purpose. If this is the case, then the knowledge of Q as a function of S might have to be obtained with few data points and be correspondingly inaccurate, particularly as Q is a rapidly-varying function of elevation, as shown by the power-laws presented above. This can degrade the accuracy of the computations considerably. In the case of the h formulation this is not a difficulty. Fenton (1989) has presented a model problem which shows this.

Although great accuracy is usually not necessary in hydrologic computations, unnecessary inaccuracies should be avoided, and certainly simpler methods should be favoured. In view of the complications and inaccuracies associated with the S formulation as described above, it seems that the h form is to be preferred. The methods described below, however, are valid for either.

3. METHODS OF SOLUTION

Whichever formulation is adopted, S or h, the problem is one of the numerical solution of a differential equation, for which there are many methods and much computer software available. Reference can be made to any book on numerical methods for a description of them. Further below, some methods are described.

The forms of the storage equation can be generalised by the expression

$$\frac{dx}{dt} = F(t,x), \qquad (6)$$

where in the case of (1), $x = S$ and $F(t,x) = I(t) - Q(t,x)$, while for (4), $x = h$ and $F(t,x) = (I(t) - Q(t,x))/A(x)$. This general form encompasses cases where the outflow is controlled and where it is not. All the following methods can be applied whether or not control is varying the form of the outflow function with time. The presence of controlled outflow is not a problem.

To commence solution it is necessary to know the initial conditions, either the value of S or h at $t = 0$, denoted by S_o and h_o respectively. At later times the subscript n is used to denote the value at time $t_n = n\Delta$ and $n+1$ at time $t_n + \Delta = t_{n+1} = (n+1)\Delta$. The time origin is taken to be at the beginning of computations.

Whichever method is used, perhaps the most complicated part of solving the differential equation is that of interpolating in a table of value pairs. It is necessary to obtain $A(h)$ for arbitrary h, or $Q(t,S)$ for arbitrary S (at a given time t if outlet is controlled). If higher order methods were to be used or time steps other than the interval of the inflow hydrograph used, it would be necessary to obtain $I(t)$ for arbitrary values of t, also from data pairs. For first and second order methods linear interpolation in a table of data pairs gives accuracy consistent with the numerical method, however this is not the case for higher-order methods, for which more accurate approximation is necessary for consistency. For reservoirs where the discharge function can have gradient discontinuities, such as where a higher spillway takes over from a lower spillway or pipe, linear interpolation in the data pairs defining the outlet characteristics would be rather more robust. Its simplicity is generally to be preferred for flood routing, where great accuracy is usually not justified.

3.1 Test problem

For purposes of comparing the accuracy and performance of different numerical methods there are some reservoir routing problems which have analytical solutions. Fenton (1989) has suggested that the following problem adapted from Yevjevich is a good one for simulating real problems. The inflow hydrograph has the same shape as a typical real hydrograph, and this problem might be adopted as a suitable test for developing computer programs to check on their accuracy or for comparing methods.

The inflow hydrograph is given by

$$I(t) = I_o + P\,t^s\,e^{-ft}, \qquad (7)$$

where P, s and f are constants defining the storm hydrograph and I_o is the base flow. The storage and area functions are given by $S(h) = a\,h^m$ and $A(h) = a\,m\,h^{m-1}$, where a and m are constants, the discharge function is $Q(h) = b\,h^m$, where b is a constant. It can be shown by rewriting the differential equation with Q as the dependent variable and solving by an integrating factor method that the problem has an analytical solution, presented by Fenton.

The problem was solved by a number of numerical methods, to examine their performance. The results shown below are those obtained from solving the h formulation of the differential equation, equation (4). The storage details correspond roughly to those of a small detention reservoir, which has a single sharp-crested weir of length about 2.5m, and the plan dimensions, if it were a square, would be about 100m by 100m if the reservoir were 2m deep. The numerical constants associated with the inflow hydrograph were chosen by trial and error to be characteristic of those encountered in practice with a single rainstorm of short duration. The time step was chosen such that the peak of the hydrograph occurred some five or six steps after commencement, a value believed to be typical. The computations were not performed using the exact analytic formula for the area function $A(h)$, but rather by interpolating in a table of ten pairs of values, as might be encountered in a practical problem where values of storage or reservoir area might only be known at a finite number of points. Similarly, the inflow hydrograph was specified by 20 numerical values of $I(t)$ obtained from (7), and interpolation used to calculate intermediate values, if necessary.

Figure 1 shows the inflow hydrograph from (7), and various numerical solutions for the outflow hydrograph. Almost all the methods described below gave results of acceptable practical accuracy.

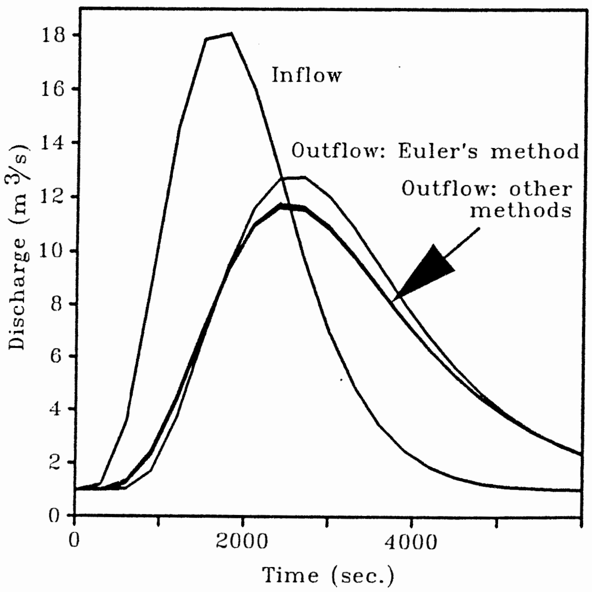

Figure 1. Results obtained by various computation schemes.

3.2 Euler's method

This is the simplest and least-accurate of all methods, being of first-order accuracy only. It is

$$x_{n+1} = x_n + \Delta\,F(t_n, x_n) + O(\Delta^2), \tag{8}$$

and the right side of the differential equation has only to be evaluated once per time step. In the case of (1) this becomes the simple expression

$$S(t+\Delta) = S(t) + \Delta\,(I(t) - Q(t, S(t))) + O(\Delta^2). \tag{9}$$

It can be seen on Figure 1 that the accuracy is not really good enough, being in error by more than 10% at the peak. However, greater accuracy can be obtained by choosing computational steps smaller than those used to define the inflow hydrograph. As this is a first-order method, taking steps 1/5 of the size would reduce the error to about 2%, and so on.

There is another advantage to using smaller time steps. Not only would it give rather more accurate results, but the peak of the hydrograph would be identified rather more accurately, an important feature. Figure 1 has been plotted using linear interpolation between computational points, and shows that without a more sophisticated means of interpolation, the peak of the hydrograph will be underestimated and for this important quantity the accuracy of the actual computational points would be rather in vain. Using smaller computational time steps the difference between the maximum computational point value and the actual peak would be reduced. An accurate method of determining the time and magnitude of the maximum from computed point values is given in Fenton (1989).

As Euler's method is so simple, and the time interval reduction such a simple and useful artifice, one is tempted to recommend this method, particularly for introducing the subject to students.

3.3 Runge-Kutta 2nd Order Method

The next of the methods in the hierarchy of accuracy is also known by several other names, including the "modified Euler method". It involves only one more function evaluation per time step and gives a scheme which is rather more accurate, and indeed of second order. The scheme is:

$$x_{n+1} = x_n + \frac{1}{2}\,(k_1 + k_2) + O(\Delta^3), \tag{10.a}$$

where

$$k_1 = \Delta\,F(t_n, x_n) \tag{10.b}$$

and

$$k_2 = \Delta\,F(t_n + \Delta, x_n + k_1). \tag{10.c}$$

This is a second order method. On Figure 1 the results define the bottom edge of the thickened line containing this and all higher-order results. The method is rather more accurate than Euler's method. The single extra evaluation of the right side of the differential equation at each step gains an order of approximation. What is convenient is that the order of accuracy is not degraded if simple linear interpolation is used to evaluate the function F at intermediate values of t and x. The results from this method on Figure 1 were obtained using linear interpolation.

3.4 Second-order implicit method

This is the method usually known as the *trapezoidal* method or the *Adams-Moulton second-order* method. Here the value of x_{n+1} is obtained from a formula in terms of itself, and the method becomes *implicit*:

$$x_{n+1} = x_n + \frac{\Delta}{2}\left(F(t_n, x_n) + F(t_n+\Delta, x_{n+1})\right) + O(\Delta^3), \quad (11)$$

and x_{n+1} appears both on the left and embedded deep in the differential equation term on the right, making this a nonlinear equation to be solved for x_{n+1}.

Placing the solution method in the context of differential equation theory shows that (11) is a consistent approximation which at worst assumes that the outflow characteristics vary uniformly between the two time levels. The case where the discharge is controlled does not necessitate an assumption that the discharge increases with no change in storage (*cf.* Laurenson, 1986). If a gate were fully opened or shut between two computational steps, which may well occur because the speed of gate opening is usually faster than the rate at which water levels fluctuate, then it might be necessary to vary the computational steps to describe that variation adequately.

Provided the time step is small enough, (11) provides a simple method of solution. Some initial estimate of x_{n+1} is obtained by Euler's method (8), and denoted here by $x_{n+1}^{(0)}$. This is substituted into the right side of (11), to give another more accurate estimate $x_{n+1}^{(1)}$ on the left. This is re-substituted on the right and the process repeated until it converges sufficiently. This is a *direct iteration* process for solving the nonlinear equation. It does not converge if the time step Δ is greater than a certain amount, but it is simply programmed.

It is noteworthy that hydrologists have tended not to use the relatively simple and direct iteration scheme (11). They have followed rather more difficult and complicated methods for solving the implicit formulation (2) which, however, converge more quickly, and whose convergence is rather more certain. Laurenson (1986) and Pilgrim (1987) have described a number of problems which can arise. Implementation of the traditional method is complicated, both at an introductory level (introducing the solution of a nonlinear transcendental equation at each time step), and more at an advanced level, where some complicated programming is necessary to treat pathological cases.

3.5 Higher-order Methods

Reference can be made to most books on numerical methods for details of these. Results are shown on Figure 1, but are almost indistinguishable from the second-order results. There seems to be relatively little point in using methods of high-order accuracy for reservoir routing. If a smaller time step were used, even Euler's method would be accurate enough.

4. STABILITY OF SOLUTION SCHEMES

To test the performance of the various solution methods for practical problems, they were applied to the sample problems included in several hydrology text and reference books, nearly all of which use the traditional S formulation. In almost all cases they performed totally satisfactorily. There were two exceptions, however, where the Runge-Kutta explicit methods gave erratic results, whereas the implicit methods performed as well as previously. One of the two examples is that in *Australian Rainfall and Runoff* (Pilgrim, 1987, page 141). The behaviour encountered in each case was that of *instability* of the explicit numerical scheme.

This was overcome by simply reducing the size of the computational time step. Halving the time step overcame most problems, and halving it again gave the results shown in Figure 2, which contains the results from all of the numerical schemes described above, showing that all were stable and sufficiently accurate. The figure is plotted with linear interpolation between computational points corresponding to the original time step as suggested by the inflow hydrograph, which is really rather large, making any computations rather demanding.

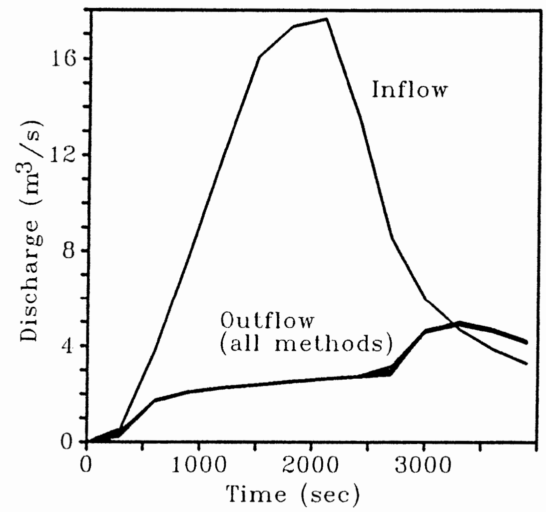

Figure 2. Results from example in *Australian Rainfall and Runoff*

Fenton (1989) showed general criteria for the stability of solution methods for reservoir routing. For a second-order Runge-Kutta scheme, using the S formulation of the storage equation, and where the discharge is not controlled, the criterion for stability becomes

$$\Delta \frac{dQ}{dS}(S) < 2, \quad (12)$$

which is convenient to test before computations commence. The table of Q and S values can be differentiated numerically, and at each point value the criterion (12) checked. If the criterion were violated, a smaller value of the time step should be used.

It is notable that the *second-order implicit* scheme (2), as used by hydrologists for many years with graphical or gradient methods of solution, is unconditionally stable. This may account for its success and continued use, despite its complexity. If the simple direct iteration process (11) were used, it would be found to be unable to converge to a solution unless the criterion (12) were satisfied.

There is a physical interpretation of the stability limit. It is that, for some incremental increase in storage level say, the change of discharge dQ must be less than twice the time rate of change of storage volume, approximated by dS/Δ. In practice this means that instability will tend to occur where spillways are wide or pipes large, changes in volume are small, or, where the computational time step proposed is too large. The sort of application where this might be a problem is that at the bottom of a detention reservoir if an outlet is placed there, where the storage volume and changes in the volume are small. This is the case presented in *Australian Rainfall and Runoff*, where an outlet pipe is located at the bottom of a detention reservoir. For more typical reservoirs the stability problem would be expected rarely to occur, as the gradient dQ/dS would be rather smaller. More often, the outlet level is some height above the reservoir bottom, so that when the spillway starts to operate, storage volume changes are very much greater, and stability is never a problem.

The easiest way of testing for stability is simply to start computing, for if a scheme is unstable this quickly manifests itself through oscillating results or numbers growing very large very quickly. If a scheme is only marginally unstable, however, this may not be noticed. A better check would be always to solve the problem for at least two different time steps and to verify that the solutions are acceptably close to each other. Usually instability is associated with inaccuracy as well, for it means that the right side of the differential equation $F(t, x)$ is varying too quickly for the time step chosen to solve the differential equation accurately. Even for implicit schemes such accuracy considerations provide a limit to the desirable time step. Laurenson (1986) provides some discussion of the mechanism by which oscillations can develop. These oscillations are a result of computing with time steps which are too large. In all cases the use of sufficiently smaller time steps renders the schemes stable and more accurate.

To conclude this section the question has to be posed: in view of the desirable stability properties of implicit methods, should they not be retained as the standard method of solution? The answer is in the negative. Their apparent complexity is a major disadvantage, yet their superior stability property is not an important advantage, for the simple artifice of reducing sufficiently the time step used in computations can always be used to satisfy the stability criteria for explicit methods in those rare cases where they are violated. Solving a flood routing problem numerically by an explicit method is usually quite trivial in terms of computer resources, so that taking smaller time steps is not a problem. Even with smaller steps it is probable that the computing resources required are smaller than with the iterative implicit method, which requires repeated evaluations at each time step.

5. CONCLUSIONS

Use of the traditional method for reservoir routing has obscured the fundamental simplicity of the problem, for it has always required some attention to mathematical and numerical details of the solution process. If, however, it is recognised that the procedure is one of solving a differential equation, then procedures can be made more simple, flexible and automatic, and more attention can be given to the real problem of reservoir routing rather than to the techniques used to solve the problem.

The traditional S formulation has the disadvantage that the storage volume and the dependence of the outflow on that volume $Q(S)$ have to be calculated. The use of the relationship between Q and S causes the accuracy of the governing equation and results to be highly dependent on the accuracy or otherwise of the numerical determination of the storage. There exists another formulation of the governing equation in terms of the reservoir surface elevation h, such that the outflow can be specified accurately and conveniently in terms of the spillway and outlet rating curves, and the storage S does not have to be calculated.

Simple explicit methods for solving differential equations are to be preferred to the traditional implicit method, which requires solution of a transcendental equation at each time step. For special cases of an extreme nature, explicit methods become unstable whereas the traditional method retains its stability. However, all the explicit methods can be made stable by reducing the computational time step, which also makes them more accurate. Further implementation of the traditional implicit method for reservoir routing should be discontinued.

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