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Inferring Wave Properties from Sub-Surface Pressure Data

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SUMMARY The problem of inferring surface data from sub-surface pressure measurements is described. A new version of local polynomial approximation is developed which is very much simpler, and is applied to several theoretical waves and to some experimental results. The new method is found to be robust and accurate, particularly so for applications where the pressure transducer is mounted higher in the water. It might have potential for the routine analysis of sub surface pressure records.

1. INTRODUCTION

A pressure transducer measures the sub-surface pressure denoted by p_n at a finite number of instants t_n, for $n = 0, 1, 2, \ldots$. The problem is then that of obtaining other wave properties from the pressure data, such as the corresponding surface elevations, the velocities in the fluid, and possibly spectra and frequency distributions of the waves and other integral quantities associated with the waves. It is surprising that so much can be inferred from a sequence of readings obtained at a single point in the fluid.

Pressure transducers have some advantages over other means of measuring wave properties. They can be located in the fluid, beneath the surface, where they might be mounted rather more securely, possibly without the need of mooring cables; they are less susceptible to the attentions of others such as fishermen; and they can be made of a rather more simple and robust construction. However the use of such pressure probes has an important disadvantage, which is that the problem of inferring free surface behaviour and velocities in the fluid from pressure records taken deep in the fluid is an ill-posed problem. Fluid motion decays exponentially down into the fluid or grows exponentially as the surface is approached. The problem is ill-posed computationally because, any fluctuations in a pressure record, from whatever source, are multiplied by exponentially large quantities when motion nearer the surface is synthesised. There has been a continuing controversy about the accuracy of the traditional spectral method for inferring surface quantities. Bishop and Donelan have given a summary of aspects of the controversy, which generated some further discussion (1).

The conventional spectral approach based on linear wave theory is to take the pressure readings, discrete Fourier transform them, use linear wave theory to find the corresponding harmonic components of the free surface elevation and velocities, and then obtain the actual velocities and surface elevations. The limiting assumption of linear theory is that all components of the waves are travelling at the speed corresponding to that phase as given by linear theory, and that there are no nonlinear interactions at all. Particularly in near-shore regions, with the observed tendency of long waves to travel as nonlinear waves of translation, where the individual components are bound to the main wave and travel with its speed, this is an unnecessarily limiting assumption. Use of the spectral method does have some further practical problems. To implement the method it is usually necessary to resort to a number of techniques of data analysis, which degrade the information provided by the original signal. These techniques include trend removal, multiplying by a "window" to remove spurious components due to end discontinuities, filtering and so on.

An alternative approach to the problem, instead of using global approximation methods such as the spectral method, where the problem is solved throughout the whole period of pressure recording, is to use local approximation methods. Nielsen (4, 5) used an approach based on local interpolation by trigonometric functions combined with linear theory. This seems to remain controversial. The method has as yet only been developed for bottom-mounted pressure transducers, whereas a modern tendency is to use buoyant cable-retained sensors rather higher in the water column, which makes very good sense in view of the poor conditioning of the computational problem.

A different approach was used by Fenton (3), who used a principle of local low-degree polynomial approximation, partly based on standard least-squares approximation methods and partly on solution locally of the full nonlinear equations of motion. By assuming polynomial variation in the vertical, the method is much less susceptible to the ill-conditioning described above. The method has the potential of being reliable and accurate. However the equations obtained were very complicated, making the method somewhat inaccessible.

This paper develops a different approach to the problem of using local polynomial approximation methods for inferring surface wave data. It is found that if instead of a local polynomial approximation for the free surface elevation, point values are used corresponding to the observational points t_n, the equations obtained are very much simpler, and a simple iterative process is suggested for the solution. When compared with numerical solutions for steadily progressing periodic waves the method is found to give good results for bottom-mounted transducers for waves longer than about eight times the water depth, and acceptable even for relatively short waves. The method is compared with some experimental results and is found to be robust and accurate. Most importantly, the simple method suggested here seems to be at least as accurate as the rather more complicated version presented by Fenton (3). More testing is necessary, yet it seems that the method might have potential for the robust and routine analysis of subsurface pressure records.

2. THEORY

It is assumed, in common with most other simple wave theories, that the waves are travelling over a flat impermeable bed, that all motion is two-dimensional, and that the fluid is incompressible and the fluid motion irrotational. In this case a velocity potential ϕ exists which is a function of time and space, such that the fluid velocity $\mathbf{u} = \nabla\phi$, and ϕ must satisfy Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \tag{1}$$

where the co-ordinate origin is on the bed, under or at the pressure probe, the x co-ordinate is in the direction of propagation of the waves and the y co-ordinate is vertically upwards.

The principle introduced in Fenton (3) and adopted in this work is that all features of the wave motion in the vicinity of the pressure probe in both space and time can be approximated locally by a relatively low-order polynomial. As the entire discussion is based on local approximation we can use a local time t, which is taken to be zero at the instant of the pressure reading. The approximation is made that the motion locally is propagating without change in the x direction with a speed c, which is as yet unknown. Hence, variation with x and t can be combined in the form $X = x - ct$, where the travelling co-ordinate X has been introduced for convenience. Locally, this is a reasonable assumption, as the time scale of distortion of the wave due to dispersion and nonlinearity is considerably larger than the local time over which the theory is required to be valid.

The local polynomial approximation is adopted such that in the vicinity of the pressure probe, throughout the depth of fluid, the velocity potential $\phi(x, y, t)$ is given by a fourth degree polynomial:

$$\begin{aligned}\phi(x,y,t) = a_0 X &+ \frac{a_1}{2}(X^2 - y^2) + \frac{a_2}{3}(X^3 - 3Xy^2) \\ &+ \frac{a_3}{4}(X^4 - 6X^2y^2 + y^4) \\ &+ \frac{a_4}{5}(X^5 - 10X^3y^2 + 5Xy^4).\end{aligned} \tag{2}$$

This equation satisfies the governing equation (1) identically throughout the flow, and it also satisfies the bottom boundary condition

$$v(x, 0, t) = \frac{\partial\phi}{\partial y}(x, 0, t) = 0. \tag{3}$$

The coefficients $a_0, ..., a_4$ are initially unknown. They are to be determined by using the information obtained by the pressure probe.

Let the pressure at the level of the probe as a function of x and t also be given by a polynomial in $X = x - ct$, corresponding to the pressure field also propagating with speed c in the positive x direction:

$$\begin{aligned}\frac{p}{\rho}(x, y_p, t) &= \sum_{j=0}^{M} p_j (x-ct)^j \\ &= R - \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial X}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]_{(x,\, y_p,\, t)} - g\, y_p,\end{aligned} \tag{4}$$

from Bernoulli's equation in the progressing frame (X, y) in which motion is presumed to be steady. R is a constant, g is gravitational acceleration, and y_p is the elevation of the pressure probe. The coefficients p_j can be found from the pressure readings $p(t_n)$ and knowledge of the speed of propagation c. Details of this are presented in the appendix.

Substituting equation (2) into equation (4) and performing the manipulations gives a power series expansion in X, and equating coefficients in like powers of X (the equation must be satisfied locally for all values of X) gives the set of equations:

$$p_4 + a_0a_4 + a_1a_3 + \frac{1}{2}a_2^2 = 0, \tag{5.1}$$

$$p_3 + a_0a_3 + a_1a_2 = 0, \tag{5.2}$$

$$p_2 + a_0a_2 + \frac{1}{2}a_1^2 + y_p^2(a_2^2 - 6a_0a_4) = 0, \tag{5.3}$$

$$p_1 + a_0a_1 + y_p^2(a_1a_2 - 3a_0a_3) = 0, \tag{5.4}$$

and

$$\begin{aligned}R = p_0 + g\, y_p + \frac{1}{2}a_0^2 &+ y_p^2\left(\frac{1}{2}a_1^2 - a_0a_2\right) \\ &+ y_p^4\left(a_0a_4 + \frac{1}{2}a_2^2 - a_1a_3\right).\end{aligned} \tag{5.5}$$

Bernoulli's equation can also be applied at the free surface, to the point over the pressure probe $x = 0$ at $t = 0$, where the surface elevation is denoted by η, usually the quantity which is sought.

$$\begin{aligned}R = g\,\eta + \frac{1}{2}a_0^2 &+ \eta^2\left(\frac{1}{2}a_1^2 - a_0a_2\right) \\ &+ \eta^4\left(a_0a_4 + \frac{1}{2}a_2^2 - a_1a_3\right).\end{aligned} \tag{6}$$

Equations (5) and (6) are a set of six nonlinear algebraic equations in the seven unknowns $a_0 \cdots a_4$, R and η. It will be seen in the appendix that to determine the pressure coefficients $p_0 \cdots p_4$ from the pressure readings, that the speed c at which the local disturbance propagates past the pressure probe also enters the equations so that there are actually eight unknowns.

The pressures in the fluid are determined by the dynamic (Bernoulli's) equation. Hence it would be expected that this equation should be solved before any others to relate pressures and surface elevations. It would be possible also to require the kinematic boundary condition on the free surface to be satisfied, and in the development of the theory this was tried. It was found that the equations which were obtained were such that solution was often difficult and erratic. Additionally it would seem a little restrictive to insist that the kinematic equation also be satisfied, in a general situation where waves of different heights and wavelengths are passing through each other.

To introduce more equations so that there are the same number of unknowns, the rather *ad hoc* assumptions are made (a) that the wave speed is given by long wave theory:

$$c = (g\,\eta)^{1/2}, \tag{7}$$

which ignores all effects of the non-uniformity of fluid velocity in the vertical, and (b), possibly even less rational is that the coefficient a_0 which is the main fluid velocity component, is simply given by

$$a_0 = -c, \tag{8}$$

such that its magnitude is equal to the speed of the disturbances, and is negative because relative to the wave the fluid is rushing backwards in the negative x direction. It is possible that a better approximation for a_0 might be discovered.

The set of nonlinear equations now can be solved. Normally solution of such a system is difficult and requires sophisticated techniques. However it was found that a very simple scheme can be used, making use of direct iteration, whereby each equation is written such that evaluating each equation allows the updating of one variable. The solution scheme is given in Table 1. The solution procedure is the nonlinear equivalent of Gauss-Seidel iteration. In the calculations reported in Section 3, it was found that the procedure worked well. It was found that convergence of this scheme was sure and rapid. Four iterations were enough to have the solution converge to four significant figures.

For each of the points in time, with an initial estimate of $\eta = p_0 + y_p$ (the hydrostatic result):

{

Calculate the coefficients P_j for $j = 0, \cdots, 4$ from equations (A.1) in the Appendix.

Iterate until converged (usually 2 iterations) by evaluating, with the initial estimate of η that from the previous point:

{

$$a_0 = -c$$
$$a_4 = -(P_4/c^4 + a_1a_3 + a_2^2/2)/a_0$$
$$a_3 = -(P_3/c^3 + a_1a_2)/a_0$$
$$a_2 = -(P_2/c^2 + a_1^2/2 + y_p^2(a_2^2 - 6a_0a_4))/a_0$$
$$a_1 = -(P_1/c + y_p^2(a_1a_2 - 3a_0a_3))/a_0$$
$$g\eta = gy_p + P_0 + (\eta^2 - y_p^2)(a_0a_2 - a_1^2/2) + (\eta^4 - y_p^4)(a_1a_3 - a_2^2/2 - a_0a_4)$$
$$c = \sqrt{g\eta}$$

}

}

Table 1. Iteration scheme for the solution for η.

It has been possible to obtain these equations by hand calculation to about sixth order without much effort. In the original formulation (Fenton, (3)), where a series expansion for η was adopted as well, the amount of calculation became prohibitive for fourth order calculations, for which it was necessary to use a symbolic algebra manipulation package on a computer. The resulting equations were very long indeed, making the method rather less accessible and dependent on the software. It is hoped that the simplicity of the equations presented here will make the method rather more useful in practice.

When computations were performed, it was found that the sequence of free-surface elevations $\eta(t_n)$ showed certain irregularities for experimental pressure data. Even though the least squares method should show smooth results, experimental variation was sufficient to show a lot of high-frequency fluctuations. It was found that a procedure could be followed such that smooth and reliable results could be had: for each point n, if a total of K points are used in the least squares approximation, then the fluid flow parameters $a_0 \dots a_4$ and equation (2) are valid over the range of all these K points, and so the surface elevation can be calculated at each of the points from Bernoulli's equation in the frame of the steady flow. In this way the final computed value might be the mean of computations at several values of n. The overall procedure can be summarised in Table 2.

For n from 0 to the end in steps of 1 or more

{

From K pressure readings symmetric about n calculate the velocity coefficients $a_0, \dots, a_4$ and the surface elevation at n, $\eta = \eta_n$, using the procedure of Table 1.

Calculate R from equation (5.5).

For j corresponding to each of the K adjacent points in turn

{

Iteratively until the process converges

{

Calculate the surface velocities

$u = \partial\phi/\partial X$, $v = \partial\phi/\partial y$

at $x = 0$, $t = t_{n+j}$ and $y = \eta_{n+j}$

from equation (2), then calculate

$$g\eta_{n+j} = R - \frac{1}{2}(u^2 + v^2)$$

}

Combine the calculated value of η_{n+j} with the previously accumulated values to give an updated mean.

}

Table 2. Overall scheme for processing a sequence of pressure readings p_n.

3. RESULTS

Comparisons were made with numerical solutions of steadily-progressing waves in water of constant depth. Solutions were obtained using a Fourier approximation method (Fenton (2)) which can accurately solve the steadily-progressing wave problem. Values of pressure at the hypothetical transducer at equally-spaced intervals were obtained, plus the corresponding surface elevations (in usual problems it is not known, of course). Using the pressure values the methods of Section 2 above were applied, as well as the more complicated version presented in (3). Quartic approximations were used throughout equation (2) and 7 adjacent points were used to fit the polynomials at each point by least squares, that is, $K = 7$.

It was to be expected that the polynomial method would work best for long waves, while for shorter waves the variation in the vertical tends to exponential and it would not be appropriate. Figure 1 shows results for a *bottom-mounted* transducer for a relatively short wave of length five times the depth ($L/d = 5$), and a Height/Depth ratio $H/d = 1/3$. The long-dashed lines are the results from various local polynomial approximation methods, including the version of (3) and the nonlinear algorithm (Table 1). Other versions will be reported in a later paper. The results are not very good, but what is notable for a wave as short as this and for this notoriously poorly posed problem, is that they are also not very wrong, showing that this might be considered a lower boundary on the wavelength for application of the theory. The inclusion of 6th degree terms in the theory would make it rather more accurate.

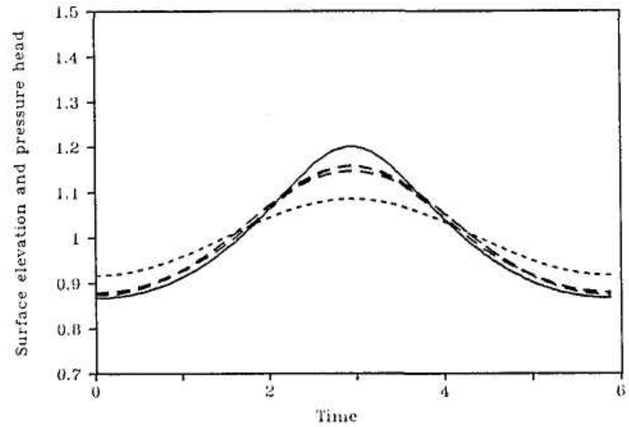

Figure 1. Actual and calculated free surface of a steady wave of height 1/3 of the depth and a length 5 times the depth, where results from a bottom-mounted transducer have been simulated. The abcissa is dimensionless time $t\sqrt{g/d}$, the ordinate shows elevation and pressure head non-dimensionalised with respect to mean depth. The solid line is the actual surface, the short-dashed line the pressure trace, and the long-dashed lines the results from various local polynomial approximation theories.

A different picture is obtained for a pressure transducer mounted at a depth of 25% of the total depth beneath the surface, for which results are shown in Figure 2, for the same wave as in the previous figure. The results are encouraging.

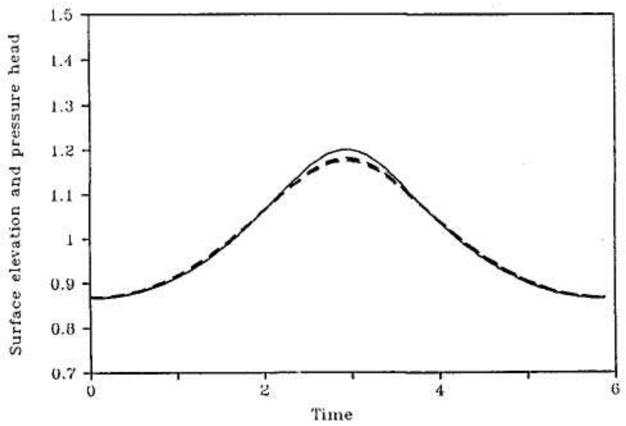

Figure 2. Results for the same wave as Figure 1, but where the simulated pressure transducer was at a depth of 25% of the total.

As expected, rather better results are obtained for longer waves. Figure 3 shows the results obtained for a wave of height $H/d = 2/3$ and a length $L/d = 15$, a wave which is both very high and long. It is a feature of the polynomial approximation method that no approximation in wave height has been introduced, unlike almost all other wave theories. Figure 3 is for a bottom-mounted transducer, and the results are quite good for this most extreme case. Results from individual theories have not been identified on the figure, but the full nonlinear theory of (3) overestimated the crest height, while the simpler theories underestimated it.

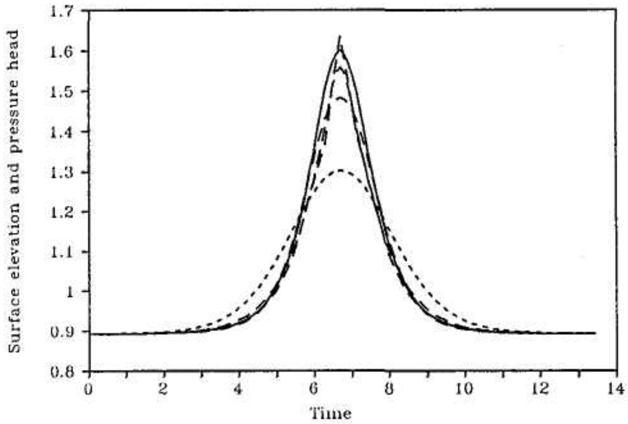

Figure 3. Results for a wave of height 67% of the depth and a length 15 times the depth, with a bottom-mounted transducer.

Figure 4 shows results for a wave of height 50% of the depth and a length of 10 times the depth, with the transducer mounted at a depth of 25%. It can be seen that results from all the polynomial approximation methods are excellent. The nonlinearity of the wave seems not to cause problems for the methods, perhaps unsurprisingly for the fully-nonlinear methods as they incorporate no height limitation.

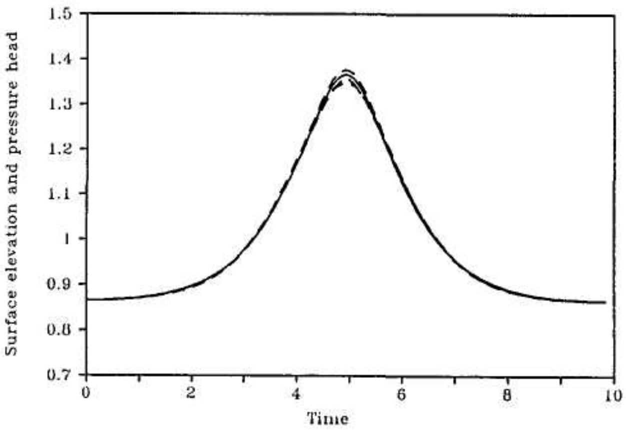

Figure 4. Results for a wave with $H/d = 0.50$, $L/d = 10$, with the simulated pressure transducer at a depth of 25% of the total.

The above results show the potential accuracy of the method, but are not a real test of the methods for actual pressure records, as the simulated pressure readings showed a very smooth variation in time. A laboratory study at Auckland University has commenced, and some of the first results are shown in Figure 5. The waves have a height of about 45% of the depth, and a dimensionless period of about 8, corresponding roughly to a length of about 8 times the depth. The waves were generated by periodic motion of a plunger-type wavemaker, and some oscillation in the tank is shown by the variation of the crest height. Water depth was about 35 cm, and pressure readings were taken from a bottom-mounted transducer at a frequency of 50 Hz. The surface elevation was measured by a capacitance wave gauge. Close examination shows that the pressure trace contains local irregularities, particularly for the last three waves. What is clear, however, is that all three versions of the polynomial approximation method generally give good results, although the trough results are not so accurate. To obtain the smooth curves shown, it was necessary to use $K = 19$ in the least squares fitting. Using results from 13 points in time gave an oscillation in the results of about 10% of the wave height. Experience gained to date suggests that the polynomial fitting method can be applied and will give results with almost any computational parameters, however the presence of oscillations shows that not enough points are being taken in the least-squares fitting to give smooth results. Once more, however, what is really notable is that the simple theories presented above seem to be just as accurate as the more formal theory reported in (3).

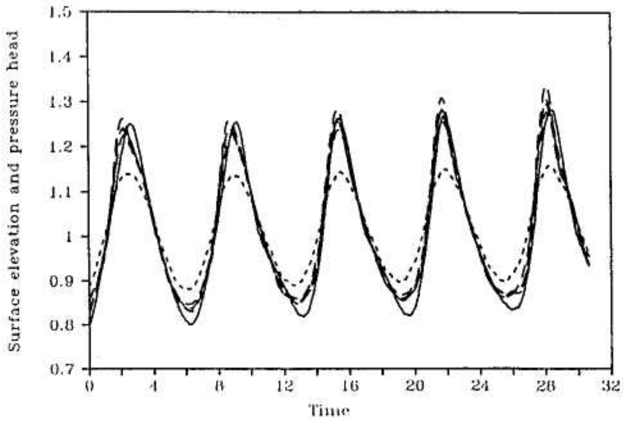

Figure 5. Experimental and computational results for a laboratory wave train. The solid line shows the experimentally-measured free surface, the long-dashed lines show results from various local polynomial methods, and the short-dashed lines show the pressure signal.

4. CONCLUSIONS

A new and much simpler version of local polynomial approximation has been developed. Preliminary results have shown it to be able to provide a robust and accurate method for the determination of wave characteristics from pressure transducer records.

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APPENDIX. Calculation of pressure coefficients

The use of a standard least-squares procedure using K points for quartic approximation was shown by Fenton (3) to give the matrix equations

$$\begin{bmatrix} K & \Sigma m^2 & \Sigma m^4 \\ \Sigma m^2 & \Sigma m^4 & \Sigma m^6 \\ \Sigma m^4 & \Sigma m^6 & \Sigma m^8 \end{bmatrix} \begin{bmatrix} P_0 \\ P_2\Delta^2 \\ P_4\Delta^4 \end{bmatrix} = \begin{bmatrix} \Sigma p_{n+m}/\rho \\ \Sigma m^2 p_{n+m}/\rho \\ \Sigma m^4 p_{n+m}/\rho \end{bmatrix},$$

and

$$\begin{bmatrix} \Sigma m^2 & \Sigma m^4 \\ \Sigma m^4 & \Sigma m^6 \end{bmatrix} \begin{bmatrix} -P_1\Delta \\ -P_3\Delta^3 \end{bmatrix} = \begin{bmatrix} \Sigma m\, p_{n+m}/\rho \\ \Sigma m^3 p_{n+m}/\rho \end{bmatrix}, \qquad \text{(A.1)}$$

in which all the summations are over K values of m, and Δ is the time step between pressure readings, K is chosen to be an odd number, and points are distributed symmetrically about the point of calculation n, such that m varies from $m = -(K-1)/2$ to $m = +(K-1)/2$. The coefficients $P_0, P_1, \ldots, P_4$ are related to the pressure coefficients by $P_j = p_j c^j$ for $j = 0, \ldots, 4$. Analytical solutions to these equations for the P_j can be written down (see any book on linear algebra). The p_j are determined as part of the iterative solution described in Section (2).