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EXPLOITING PERIODICITY IN BOUNDARY INTEGRAL EQUATIONS

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ABSTRACT: *The numerical approximation of boundary integral equations has traditionally used low-order algebraic methods, which lead to long and complicated formulae. If it is recognised that in circumnavigating the boundary of a computational region everything is periodic, the approximation of the integral can be done by methods which are considerably more accurate, but also remarkably simpler, although some effort must be made to ensure continuity by the distribution of computational points. Results obtained demonstrate the accuracy of the procedure. The methods described here in detail are for potential problems in two dimensions only, however they are capable of extension to other problems and into three dimensions.*

1. INTRODUCTION

An advantage of the use of boundary integral equations is that, instead of the whole region, only the boundary has to be discretised for numerical solution, and boundaries of quite irregular shape can be treated. However, the numerical approximation of the boundary and the field variable becomes a demanding task analytically. Traditionally low-order polynomial approximation methods have been used, which are accessible, and at lowest order are not complicated. Neither are they particularly accurate. For increasing accuracy, higher-order methods are used, and these become very complicated indeed (Brebbia and Dominguez, 1989, for example).

This paper attempts to show how periodicity may be exploited. As the boundary of a two-dimensional region is circumnavigated, the problem is actually periodic, for as a second traverse is made, both the geometry and the field variables are the same as in the first, and so on. It is possible to exploit this periodicity of the problem to give a method which is capable of greater accuracy than traditional methods, yet is much simple. In practice, some effort has to be made to ensure sufficient continuity of the integrand as well. In this paper the singularity of the kernel in the Cauchy integral theorem is eliminated, such that the integrand is everywhere continuous. Also, to ensure continuity of the integrand if arcs of the boundary intersect at an angle at any point, it is necessary to space computational points such that sufficient continuity is maintained.

The system of linear equations which results from the numerical approximation is diagonally-dominant, such that iteration methods can be used for numerical solution, which allows huge numbers of points can be used.

Results presented demonstrate the accuracy and convenience of the method, especially for the subsequent computation of field values, where the traditional method is shown to be badly conditioned.

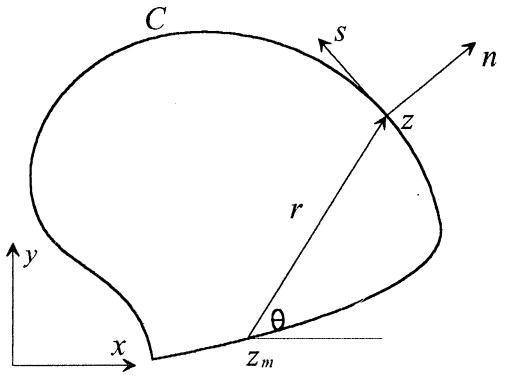

Figure 1: Typical geometry of boundary, showing important co-ordinates

2. THEORY

Consider a two-dimensional region such as that shown in Figure 1 in which a scalar potential function ϕ exists and satisfies Laplace's equation, $\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0$. A typical boundary value problem is where the value of ϕ or its normal derivative $\partial\phi/\partial n$ or a combination of the two is known at all points on the closed boundary C. The usual way of doing this is to set up and solve the integral equation:

$$\alpha\phi(z_m) = \int_C \left(\frac{\phi}{r}\frac{\partial r}{\partial n} - \frac{\partial \phi}{\partial n} \log r \right) ds, \qquad (1)$$

where s is an arc-length co-ordinate around the contour, α is the interior angle at the point m, which if the boundary turns continuously is π, r is the distance of a general point from m, and at the general point a normal and tangential co-ordinate system (s, n) exists. Equation (1) is the form which has received most attention in the literature, for example Liggett and Liu (1983), Brebbia(1984), although an equivalent complex form has been used (Hunt and Isaacs, 1981). Whichever form is used, numerical approximation of the integral near the singularity at m is demanding, and a great deal of effort has to be given to the details of computation schemes. For example, the second order scheme described in Brebbia and Dominguez (1989) approximates the integrands allowing quadratic variation and a great deal of complicated mathematics has to be worked through and presented. For higher orders of approximation the effort would be prohibitive.

It is the aim of this paper to develop a formulation of the problem in which the integrands are not singular, and more

importantly to exploit the fact that the problem is periodic as one traces around the boundary. In this way, apparently paradoxically, numerical schemes can be developed which have the advantages of being both simple and accurate.

Consider a typical problem where ϕ and its normal derivative $\partial\phi/\partial n$ are specified at different parts of the boundary. The problem is to determine ϕ within and on C. Here, the problem is set in the context of complex functions; it will be seen that some advantages accrue.

As ϕ is an harmonic function, another function ψ exists, and the complex function $w=\phi+i\psi$, where $i=\sqrt{-1}$, *Cauchy's theorem* states that w satisfies

$$\oint w(z)\,dz = 0 \qquad (2)$$

where the path of integration is around the contour C.

In this work, an approach is adopted in which the singularity is subtracted. This can be easily done, for if $w(z)$ is analytic within and on C, then so is the function $(w(z)-w(z_m))/(z-z_m)$, and equation (2) can be written with this as integrand:

$$\oint \frac{w(z)-w(z_m)}{z-z_m}\,dz = 0\ , \qquad (3)$$

where the integrand is everywhere continuous, even at $z=z_m$, and its numerical approximation should be simpler and potentially more accurate.

3. NUMERICAL SCHEME WITH BOUNDARY PERIODICITY

A feature of boundary integral methods which seems not to have been exploited is the fact that around the boundary, all variation is periodic, for in a second circumnavigation of the boundary, the integrand is the same as in the first, and so on. This suggests the use of methods which exploit periodicity to gain handsomely in accuracy. A continuous co-ordinate j is introduced here, which is 0 at some reference point on the boundary, and after a circumnavigation of the boundary, takes on a value N, which will be taken to be an integer. The integral in equation (3) can be written

$$\int_0^N \frac{w(z(j))-w(z_m)}{z(j)-z_m}\frac{dz}{dj}\,dj = 0\ . \qquad (4)$$

Now the numerical approximation is introduced to transform the integral equation into an algebraic one in terms of point values. To do this, the integral in equation (4) is replaced by the trapezoidal rule approximation:

$$\sum_{j=0}^{N-1} \frac{w(z_j)-w(z_m)}{z_j-z_m} z_j' = 0\ , \qquad (5)$$

where $z_j = z(j)$ and $z_j' = dz(j)/dj$, but where, after the differentiation, j now takes on only integer values. It might be thought undesirable that the usually low-accuracy trapezoidal rule has been used, however, where the integrand is periodic, as it is here, the trapezoidal rule is capable of very high accuracy indeed. If the integrand is periodic and has a continuous kth derivative, and if the integral is taken over a period, then (Abramowitz and Stegun, 1965, #25.4.3):

$$\text{Error} \le \frac{\text{Constant}}{N^k}.$$

For functions which are of low degrees of continuity, where k might be 2, 3 or 4 say, the accuracy will be comparable to traditional low-level polynomial approximation of the integrals, however if high degrees of continuity exist, the method should be very accurate indeed. Indeed, the error is of the order of magnitude of the last term in a Fourier series with $N/2$ terms which interpolates the N points. Throughout this work, the accuracy attained is this "Fourier" accuracy.

It is interesting that equation (5), which required little effort in its derivation and which has remarkably simple weights for the $w(z_j)$, is a numerical approximation to the original integral which may be very much more accurate than conventional approximations which assume particular variations of the unknown (ϕ or w) and of the boundary, whether constant, linear or quadratic over an "element", and which may use high-order Gauss formulae for the integrals. Such calculations are long and tedious, as are the formulae obtained (Brebbia and Dominguez, 1989). The present formulation seems to be rather simpler, and has no need for the concept of boundary elements, as it uses the points on the boundary as mere interpolation points.

In the form of equation (5), the expression is not yet useful, as the case at the singularity $j=m$ has to be addressed. It is easily shown that in this limit, the integrand and hence the summand becomes $dw(m)/dm$, and extracting this term from the sum gives the expression in terms of a "punctured sum" $j\neq m$:

$$\frac{dw}{dm}(m) + \sum_{j=0,\,j\neq m}^{N-1} \frac{w_j-w_m}{z_j-z_m} z_j' = 0, \qquad (6)$$

for $m=0,\ 1,\ 2,\ \ldots,\ N-1$, and where the obvious notation $w_j = w(j)$ *etc.* has been introduced. The notation $dw(m)/dm$ means differentiation with respect to the continuous variable m, evaluated at integer value m. It is convenient here to introduce the symbol Ω_{mj} for the geometrical coefficients:

$$\Omega_{mj} = \alpha_{mj} + i\beta_{mj} = \frac{z_j'}{z_j-z_m},$$

with the real geometrical coefficients α_{mj} and β_{mj} thus defined. One is free to use either the real or imaginary part of the integral equation and of the sum, equation (6). Here the two parts are extracted, to give

$$\frac{d\phi}{dm}(m) + \sum_{j=0,\,j\neq m}^{N-1} \left[\alpha_{mj}(\phi_j-\phi_m) - \beta_{mj}(\psi_j-\psi_m)\right] = 0, \qquad (7)$$

$$\frac{d\psi}{dm}(m) + \sum_{j=0,\,j\neq m}^{N-1} \left[\alpha_{mj}(\psi_j-\psi_m) + \beta_{mj}(\phi_j-\phi_m)\right] = 0. \qquad (8)$$

Provided either $d\phi/dm$ or $d\psi/dm$ is known at every point on the boundary, one of these equations can be used at each of the N computational points, written in terms of the $2N$ values of ϕ_j and ψ_j. If N of these are known, specified as boundary conditions, then there are enough linear algebraic equations and it is possible to solve for all the remaining unknowns.

4. BOUNDARY CONDITIONS

There are two common boundary conditions:

1. Dirichlet conditions

These are when ϕ or ψ is specified along an arc, a term used here to denote part of the boundary. If the computational point spacing is known, such that $dz(m)/dm$ can be calculated, then for example, $d\phi(m)/dm$ in equation (7) can be calculated from

$$\frac{d\phi}{dm}(m) = \frac{\partial\phi}{\partial s}\frac{ds}{dm} = \frac{\partial\phi}{\partial s}\left|\frac{dz}{dm}\right|, \qquad (9)$$

where s is a boundary arc-length variable with arbitrary origin, and where ϕ is known along an arc such that $\partial\phi/\partial s$ can be calculated. A common occurrence is where ϕ or ψ is constant along a boundary, in which case the corresponding $d\phi/dm$ or $d\psi/dm$ is zero, and the implementation is particularly simple.

2. Neumann conditions

These occur where the normal derivative $\partial\phi/\partial n$ is specified along an arc such that $\partial\phi/\partial n = f(s)$, where $f(s)$ is known, then

using one of the Cauchy-Riemann equations gives $\partial\psi/\partial s = f(s)$ and integrating gives $\psi = F(s) + C$, where C is an arbitrary constant of integration. For computational purposes, this is written $\psi_m = \Psi(m) + C_n$, where $\Psi(m)$ is a known function of m, and C_n is a constant on this, the nth Neumann boundary arc. It can be shown that Neumann boundaries can be treated by methods similar to those described above for Dirichlet conditions.

5. DISTRIBUTION OF COMPUTATIONAL POINTS

The linear algebraic equations equivalent to the integral equations have now been expressed in terms of the co-ordinates of the computational points z_j and the derivatives around the boundary, z_j'. The accuracy of the method depends on how continuous the latter are, and it is necessary to spend some effort in ensuring continuity across corners in the boundary, something which conventional methods which use local approximation do not have to treat.

Consider an arc of a boundary which may have a corner at each end. Let there be some parameter such as arc length s, which uniquely specifies position on the arc. If ds/dj is continuous along the arc *and across the corners at its ends*, then so is z_j'. The problem is to distribute the computational points such that ds/dj is continuous across the ends. It seems that the easiest way of doing this is to choose the points such that ds/dj and possibly several higher derivatives go to zero at the ends. If this is done for all arcs of the boundary, there will be a high degree of continuity at each corner. This was implemented and found to work very well.

6. NUMERICAL COMPUTATION OF COEFFICIENTS

In some situations it might be considered too tedious to compute the z_j' explicitly, or it might even not be possible, such as for irregular boundaries. It is convenient to have a numerical method which can treat arbitrary geometries to a similar level of accuracy as the rest of the computations. This can be done using Fourier approximation, and a simple expression can be obtained using Fourier approximation methods. It is:

$$z_j' = -\frac{\pi}{N} \sum_{n=0,\, n\neq j}^{N-1} z_n (-1)^{j-n} \cot \frac{\pi(j-n)}{N},$$

for $j = 0, \ldots, N-1$, which is easily programmed. The computational cost is $O(N^2)$, but the accuracy is that of the rest of this work, which is Fourier accuracy, and solution methods typically are of this computational cost anyway. A more economical method would be to use discrete Fourier transforms to evaluate these derivatives, but this step is not an important part of the overall computational cost, and using Fourier methods would require the provision of such programs.

7. SET-UP AND SOLUTION OF EQUATIONS

When the z_j' have been calculated, the coefficients α_{mj} and β_{mj} are known, and can be used in equations (7) and (8), one for each point at which an unknown exists. There is a certain amount of good fortune here, for if the magnitudes of the coefficients are examined, it may be seen that the system of equations is such that the equation written for point m is dominated by the coefficient of the unknown at that point. This suggests that the iterative procedure of Gauss-Seidel is a possibility, which was found to work well. It was found that a lot of programming detail could be avoided if the step of assembling into a matrix was by-passed. In this case, the equations (7) and (8) may simply be written in terms of the appropriate dominant unknown on the left side and sequentially evaluated. It was found that huge numbers of points could be used if the coefficients of each equation were generated at the time the equation was evaluated. The implementation of the scheme in this equation-by-equation iterative form was particularly simple.

8. POST-PROCESSING: CALCULATION OF VALUES

After a solution has been obtained it is usually necessary to be able to calculate values of ϕ or w at an arbitrary point, z_n say. In this case, it is no longer necessary to take the limiting procedure, and (5) can be written as

$$\sum_{j=0}^{N-1} (w_j - w_n)\Omega_{nj} = 0. \qquad (10)$$

Provided $w(z)$ is sufficiently continuous, this is an excellent approximation to the original integral equation (3). Separating the components gives:

$$w_n = \frac{\sum_{j=0}^{N-1} w_j\, \Omega_{nj}}{\sum_{j=0}^{N-1} \Omega_{nj}}, \qquad (11)$$

from which it is simple to extract either the real or the imaginary part.

9. RESULTS

The test problem which was chosen for comparing results from the present work with traditional polynomial approximation methods was that of the St Venant torsion of a square bar (Brebbia and Dominguez, 1989). The problem can be stated in mathematical terms: $\nabla^2\phi = 0$ within a rectangle of x dimension $2a$ and y dimension $2b$. It is possible to exploit symmetry so that we can just consider a quarter of the rectangle, and in this work we consider a square with $a = b = 1$, such that $0 \le x \le 1$ and $0 \le y \le 1$, with the boundary conditions: $\phi = 0$ on the two sides $y = 0$, $0 \le x \le 1$ and $x = 0$, $0 \le y \le 1$, while $\partial\phi/\partial n = y$ on $y = 1$, $0 \le x \le 1$, and $\partial\phi/\partial n = -x$ on $y = 1, 0 \le x \le 1$. This problem has an analytical solution which can be used for comparison purposes.

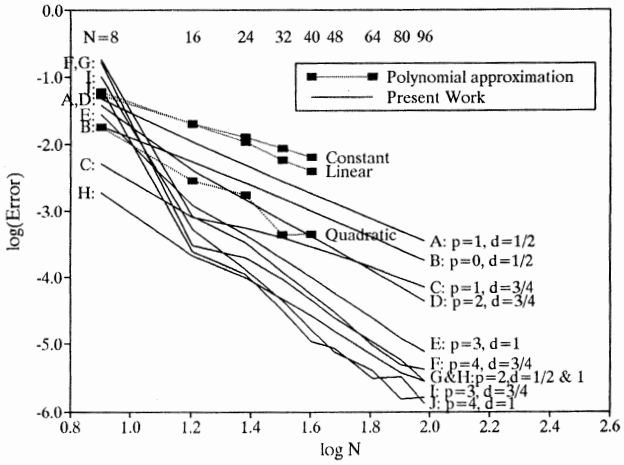

Figure 2: Accuracy comparison between polynomial approximation and the present work

Figure 2 shows the results obtained by using three conventional boundary element computer programs (Brebbia and Dominguez, 1989) which use a hierarchy of approximations based on polynomial approximation, successively assuming that quantities are (1) constant over an element, (2) vary linearly across an element and (3) vary quadratically across two

elements. Results from the present method are shown for comparison. The parameter p refers to the degree of continuity across corners of the square ($p = 0$ means that the zeroth derivative was continuous, giving equally-spaced points, and so on). The results do not throw much light on the optimal values of the end-spacing parameter d, which will not be described in this paper. What is clear, however, is that the present method is capable of rather greater accuracy than the polynomial approximation method using boundary elements. For few computational points, $N = 8$, the traditional method was more accurate - reflecting the fact that an 8-term Fourier series approximation to a square is not a very good approximation. For larger values of N, however, the present work was capable of considerably greater accuracy, especially for large numbers of points.

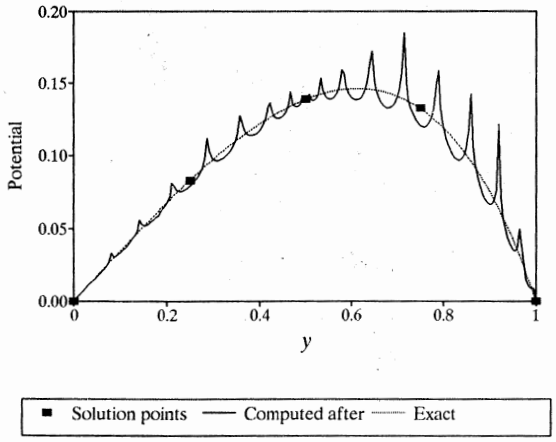

Figure 3(a): Potential distribution obtained from polynomial approximation program

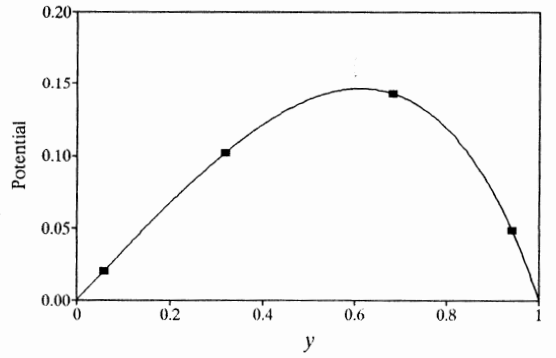

Fig. 3(b): Potential distribution from the present work (exact results obscured by computational results)

It is not until the details of the field solution are computed, that some of the advantages of the present method become apparent. In Figure 3(a) are the computational results from the three traditional boundary element methods for the distribution of ϕ along the face of the square: $x = 1$, $0 \le y \le 1$, using a total of 16 computational points. It can be seen that the computed point values agree closely with the exact solution. When, however, the value of ϕ is computed at points along the boundary the results are very poor indeed. It seems that the singular nature of the integrals used in that method give large fluctuations as computational singularities are traversed around the boundary (there are 16 of the little peaks, equal to the number of points). In Figure 3(b) results from the present work are shown, using equation (11). The accuracy of that expression, obtained via an implicit singularity subtraction, is clearly very high indeed, and there seems to be no numerical ill-conditioning. The method did fail if a computational point ever coincided to machine accuracy with one of the points used in the solution, however, provided it was not coincident there was no apparent ill-conditioning at all, a rather surprising result. For example, as close as 10^{-11} to any computational point, the computed solution was still accurate to the full machine accuracy, some 14 figures.

The implications of this are important, for if field values cannot be computed at an arbitrary point to acceptable accuracy, then achieving accuracy for the computational points themselves is somewhat in vain. For example, any contour plot of the field would be very ragged. Using the present approach, however, enables contour plots of high accuracy to be produced. In Figure 4, the boundary geometry and points for a square are shown, as well as the computed field for the demanding case where there is a logarithmic singularity at the corner of the flow field, corresponding to a source or sink there. In this case the values of the potential function become infinite at the singularity, and the function ceases to be analytic, thereby violating one of the assumptions on which the present numerical method is based. However, as can be seen in the figure, the method actually can solve this problem. To obtain a solution which when plotted was almost visually ideal it was necessary to use as many as 96 points. With fewer points, the contours showed some irregularities, not unexpected when it is realised that the path of integration actually passes through the singularity. As could be inferred from the figure however, it would seem that the boundary, when approximated by the Fourier series, passes just to one side of the singularity, and it is not included inside the boundary. The success of the method in this case is important, for such singularities are routinely encountered, even where the boundary conditions may appear innocuous.

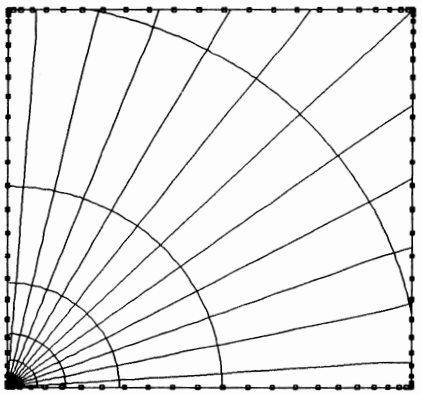

Figure 4: Flow field due to a source (logarithmic singularity) at the bottom left corner

10. REFERENCES

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