Simulating wave shoaling with boundary integral equations

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SUMMARY A numerical method is developed for the solution of wave propagation problems. Like several others, it makes no essential analytical approximation. Unlike the others, it is simple to program and potentially rather more accurate. The results presented in this paper show that it is capable of accurately modelling problems of nonlinear wave propagation and shoaling and some detailed results are presented. However, in the present formulation its performance was not as robust as would be expected from a routine tool, and more development seems necessary, for which some suggestions are made.

1. INTRODUCTION

Numerical methods for wave propagation using integral equations have been proposed for some years. These almost always require the solution of the governing field equation at each step in time, which if the fluid is assumed incompressible and the flow irrotational, is Laplace's equation, which makes computations demanding. Usually the solution is by means of a boundary integral equation, either obtained using a Green's function, or using Cauchy's integral theorem. Then, having solved for the potential distribution around the boundary, the velocity of a point on the boundary may be calculated, and Bernoulli's equation used to calculate the rate of change of the potential on the boundary. These give differential equations for each boundary point, and the solution and the wave may be advanced in time. Usually it is not necessary to make any essential analytical approximation, such as is otherwise usually the case in studies of water waves.

This approach was initiated by Longuet-Higgins and Cokelet (1) for the study of waves in deep water. Some of the methods used a Green's function method, which set up and solved a boundary integral equation with a logarithmic kernel. A different approach was introduced by Vinje and Brevig (2), who used the Cauchy integral theorem in terms of a complex potential function as the integral equation valid around the boundary .

A brief history of various attempts using these methods is described by Liu *et al.* (3). An interesting different approach was introduced recently by Leitao and Fernandes (4), who took as the upper surface of the computational domain the undisturbed water surface, solved Laplace's equation in that region, but used a second-order Taylor expansion on the undisturbed surface instead of applying the exact boundary conditions on the actual surface. In this way, their computational domain was constant, and they only had to solve a matrix equation once, rather than at each time step. The process of timestepping following the evolution of the surface waves then simply involved using the values on the undisturbed surface. However, only a second order set of results could be obtained.

It is the impression of the author that the methods, except for those which confine themselves to periodic waves over a horizontal or infinitely deep bed which can then use a very concise computational domain, (1) or (5) for example, are still not accurate enough to be considered for use as a reliable tool by coastal engineers.

The author (6) has developed a method for the solution of Laplace's equation in two dimensions which has some advantages over traditional methods: it is simpler in theory and implementation, yet is more accurate, and of particular importance to the problem of shoaling waves, it is computationally robust and allows the use of iterative methods for solution. It is the aim of this paper to apply that method to the problem of shoaling waves. In the following sections the theory and computational methods are presented, then application to some problems of wave propagation and shoaling is described. It will be seen that even the method advocated, despite its formidable accuracy for fixed domains, is somewhat fragile when the domain is allowed to move, such as for a free surface problem. Some results are presented, showing how it can solve wave propagation accurately, but in practice it was found to be not as robust as the author had hoped, and it has not yet been able to simulate the overturning of a plunging breaker. Nevertheless, as it is simpler than other approaches, it may be preferred. Its real *metier* may yet prove to be the approach of Leitao and Femandes, where a single highly accurate solution on a fixed domain is required.

2. THE INTEGRAL EQUATION

Consider a two-dimensional region such as that shown in Figure 1 containing an incompressible fluid which flows irrotationally, in which case a scalar potential function <)> exists and satisfies Laplace's equation: V^2(j> = 0. A typical boundary value problem is where the value of <|> or its normal derivative *dtydn* or a combination of the two is known at all points on the closed boundary C. One way of doing this, in which only values on the boundary have to be considered, is to solve the integral equation:

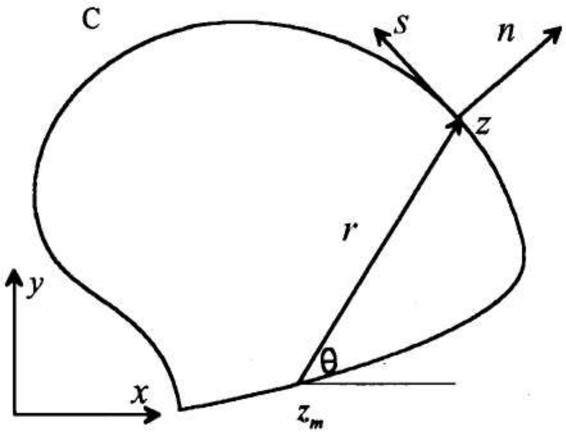

Figure 1. Typical computational domain showing important points and coordinates

$$\pi_m \phi_m = \int_C \left(\frac{\phi}{r}\frac{\partial r}{\partial n} - \frac{\partial \phi}{\partial n} \log r \right) ds \tag{1}$$

where s is an arc length co-ordinate around the contour, π_m is the interior angle at the point m, (which if the boundary is smooth at that point is π), r is the distance of the general point from m, and at that point a normal and tangential co-ordinate system (s, n) exists. If necessary, the integrals are to be interpreted in a principal value sense. Equation (1) is the form that has received most attention in the literature . However, numerical approximation of the integrals and the boundary is demanding, especially near the singularity at m, and considerable effort has to be given to the details of computation schemes. For example, to approximate the integrands and the boundary by quadratic variation, a great deal of complicated mathematics has to be worked through and presented, and for higher orders of approximation the effort would be prohibitive.

In Reference (6), a complex integral equation is derived, from which equation (1) can be obtained. The complex equation is, however, not singular and is easier to approximate numerically for a given level of accuracy. The computational points may be interpolated by Fourier series for plotting purposes, and a simple scheme is developed to compute the weighting coefficients for a completely general shape of figure. The theory is briefly restated here: As ϕ is an harmonic function, another function ψ exists, a conjugate harmonic function, related to ϕ by the Cauchy-Riemann equations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{2}$$

It can be shown that if these relations are satisfied, then the complex function $w = \phi + i\psi$, where $i = \sqrt{-1}$, has a unique derivative with respect to the complex variable $z = x + iy$, and the Cauchy integral theorem is valid and can be written with $(w(z) - w(z_m))/(z - z_m)$ as integrand:

$$\oint \frac{w(z) - w(z_m)}{z - z_m} dz = 0. \tag{3}$$

In this equation the integrand is everywhere continuous, even at $z = z_m$, and its numerical approximation should be simpler and potentially more accurate than in equation (1). In this form it is not necessary to calculate the angle π_m at each point. It will be shown below that using the stream function ψ and equation (3) leads to a system of algebraic equations which are all nearly diagonally dominant, unlike equations obtained from equation (1).

3. BOUNDARY CONDITIONS

Solid Boundaries

In this paper which is primarily concerned with the shoaling of waves we will not consider the problem where part of the solid boundary might move, such as in wavemaker problems or the generation of tsunami. On the sea bed, assumed impermeable here, the condition that flow does not cross the boundary is simply that, on the boundary,

$$\psi = 0. \tag{4}$$

Free Surface

On the free surface, denoted by $y = \eta(x, t)$, the governing equations are nonlinear, partly because the location of the free surface also appears in them. There are two equations: one is the kinematic condition that the velocity of a particle on the surface is equal to the fluid velocity at that point. Thus, if x_m and y_m are the coordinates of a point on the surface, then

$$\frac{dx_m}{dt} = \frac{\partial \phi}{\partial x}(x_m, y_m, t) \quad \text{and} \quad \frac{dy_m}{dt} = \frac{\partial \phi}{\partial y}(x_m, y_m, t), \tag{5}$$

or, using the fact that the complex conjugate of the fluid velocity is given by the derivative $dw/dz = u - iv$:

$$\frac{d(x_m - iy_m)}{dt} = \frac{dw}{dz}(z_m, t), \tag{6}$$

where the real and imaginary parts provide a pair of ordinary differential equations for the movement of the surface particle.

The second free surface boundary condition is obtained

(a) from the pressure equation (the unsteady Bernoulli equation):

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + g\eta_m + \frac{1}{2}\left|\frac{dw}{dz}\right|_m^2 = 0, \tag{7}$$

where p/ρ is the pressure divided by the fluid density, which is zero on the surface, $y_m = \eta_m$ at the surface particle and g is the gravitational acceleration, and

(b) from the expression for the rate of change of ϕ at a particle, the *material* derivative:

$$\frac{D\phi_m}{Dt} = \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial t} + u^2 + v^2 = \frac{\partial \phi}{\partial t} + \left|\frac{dw}{dz}\right|_m^2, \tag{8}$$

which, setting $p = 0$ on the surface, combine to give

$$\frac{D\phi_m}{Dt} = -g\eta_m + \frac{1}{2}\left|\frac{dw}{dz}\right|_m^2, \tag{9}$$

a differential equation for ϕ_m at the free surface particle. This gives us a way of calculating ϕ as time evolves so that at any instant it is known at all points on the free surface, while we know that $\psi = 0$ on the sea bed from equation (4). Hence we have enough boundary information to obtain a solution of equation (3) at each time step, namely to obtain the values of ϕ on the bottom and ψ on the free surface, so that $\omega = \phi + i\psi$ is known at all points, dw/dz can be calculated, the solution advanced, and so on.

4. NUMERICAL SCHEME USING PERIODICITY AROUND THE CONTOUR

A feature of boundary integral methods exploited in reference (6) is that around the boundary all variation is periodic, for in a second circumnavigation of the boundary the integrand is the same as in the first, and so on. This suggests the use of methods that exploit periodicity to gain handsomely in accuracy. A continuous co-ordinate j is introduced here, which is 0 at some reference point on the boundary, and after a complete circumnavigation of the boundary has a value N, which will be taken to be an integer. The integral in equation (3) can be written

$$\int_0^N \frac{w(z(j)) - w(z_m)}{z(j) - z_m} \frac{dz}{dj} dj = 0. \tag{10}$$

Now a numerical approximation is introduced to transform the integral equation into an algebraic one in terms of point values. The integral in equation (10) is replaced by the trapezoidal rule approximation:

$$\sum_{j=0}^{N-1} \frac{w(z_j) - w(z_m)}{z_j - z_m} z_j' = 0, \tag{11}$$

where $z_j = z(j)$ and $z_j' = dz(j)/dj$, but in which after the differentiation, j takes on only integer values. In this case the trapezoidal rule has reduced to the simple sum as the end contributions are from the same point, $z_0 = z_N$ because of the periodicity. This is a particularly simple scheme when compared with some such as Gaussian formulae which have been used to approximate boundary integrals. Where the integrand is periodic, as it is here, the trapezoidal rule is capable of very high accuracy indeed. If it is periodic and has a continuous kth derivative, and if the integral is over a period, then:

$$\text{Error} \le \frac{\text{Constant}}{N^k}. \tag{12}$$

For functions that are of low degrees of continuity, where k might be 0, 1 or 2 say, the accuracy will be comparable to traditional low-level polynomial approximation of the integrals, however if high degrees of continuity exist, the method should be very accurate, as was shown in reference (6).

In the form of equation (11), the expression is not yet useful, as the point $j = m$ has to be considered. It is easily shown that in this limit, the integrand (and hence the summand) becomes $dw(m)/dm$, and extracting this term from the sum gives the expression with a "punctured sum" $j \ne m$:

$$\frac{dw}{dm}(m) + \sum_{j=0,\, j\ne m}^{N-1} \frac{w_j - w_m}{z_j - z_m} z_j' = 0, \tag{13}$$

for $m = 0, 1, 2, \ldots, N-1$, and where the obvious notation $w_j = w(j)$ *etc.* has been introduced. The notation $dw(m)/dm$ means differentiation with respect to the continuous variable m, evaluated at integer value m. It is convenient here to introduce the symbol Ω_{mj} for the geometric coefficients:

$$\Omega_{mj} = \alpha_{mj} + i\beta_{mj} = \frac{z_j'}{z_j - z_m}, \tag{14}$$

whose real and imaginary parts are the coefficients α_{mj} and β_{mj}. Equation (13) becomes

$$\frac{dw}{dm}(m) + \sum_{j=0,\, j\ne m}^{N-1} \Omega_{mj}(w_j - w_m) = 0. \tag{15}$$

It is easily shown, writing $z(j)$ in complex polar notation as $z(j) = z_m + r(j)e^{i\theta(j)}$, that

$$\alpha_{mj} = \frac{1}{r}\frac{dr}{dj} = \frac{d}{dj}(\log r), \quad \text{and} \quad \beta_{mj} = \frac{d\theta}{dj}, \tag{16a,b}$$

such that

$$\Omega_{mj} = \frac{d}{dj}(\log\,(z_j - z_m)), \tag{16c}$$

also able to be obtained from equation (14). These weighting coefficients can be seen to be related to kernels of the real integral equation (1). They have a relatively simple physical significance.

One is free to use either the real or imaginary part of the integral equation and of the sums which approximate it, equation (13) or (15). The two parts can be extracted to give

$$\frac{d\phi}{dm}(m) + \sum_{j=0,\, j\ne m}^{N-1} [\alpha_{mj}(\phi_j - \phi_m) - \beta_{mj}(\psi_j - \psi_m)] = 0, \tag{17a}$$

and

$$\frac{d\psi}{dm}(m) + \sum_{j=0,\, j\ne m}^{N-1} [\alpha_{mj}(\psi_j - \psi_m) + \beta_{mj}(\phi_j - \phi_m)] = 0. \tag{17b}$$

One of these equations can be used at each of the N computational points, provided either $d\phi/dm$ or $d\psi/dm$ is known that point, which can be done from the boundary conditions as described above. Each equation is written in terms of the $2N$ values of ϕ_j and ψ_j . If N of these are known, specified as boundary conditions, then there are enough linear algebraic equations and it should be possible to solve for all the remaining unknowns.

It can be shown that in these equations, the dominant coefficients are the sum $\Sigma_{j=0 j\ne m}^{N-1} \beta_{mj}$, the coefficient of ψ_m in (17a) and ϕ_m in (17b), and it can be shown that the equations are nearly diagonally dominant in those quantities. This is fortunate, for as equation (17a) can be used on the free surface where $d\phi/dm$ can be evaluated and where ψ_m is the unknown and (17b) on the sea bed where $d\psi/dm = 0$, and where ϕ_m is unknown, the system of equations is nearly diagonally dominant, which suggests a certain computational robustness, and the possibility of iterative solution.

5. DISTRIBUTION OF COMPUTATIONAL POINTS

The linear algebraic equations approximating the integral equations have been expressed relatively simply in terms of the coordinates of the computational points z_j and the derivative around the boundary, z_j'. The accuracy of the method depends on how continuous the latter are, and in Reference (6) some effort was spent in ensuring continuity across corners of the boundary. In fact it was found that even if no special spacing was used, the accuracy was still surprisingly high. In the original paper some effort went into producing a system capable of exploiting fully the potential accuracy of the method. When it was applied in the present work to moving boundary problems the disappointing result was obtained that as the boundary points moved, the most sophisticated schemes for point spacing became the most inappropriate, as the accuracy of the scheme was quickly destroyed by the movement of the points. In this work it was found that the most robust schemes obtained using simply equally-spaced points.

6. NUMERICAL COMPUTATION OF COEFFICIENTS

In problems of wave shoaling, the boundary of the computational region, including the sea bed and the free surface, is quite irregular. The periodicity around the boundary may be exploited to give a simple scheme for computing the necessary derivatives around the boundary. The main problem is to compute values of the z_j'. Also, it is convenient to be able to use a means of interpolation between the computational points for plotting purposes which has the same accuracy as the underlying numerical method. Both can be accomplished simply and economically using Fourier approximation.

Suppose the position of each of the N boundary points z_j, $j = 0, 1, \cdots, N-1$, is known. Consider the discrete Fourier transform of the points:

$$Z_m = \frac{1}{N}\sum_{j=0}^{N-1} z_j e^{-i2\pi mj/N} = D(z_j\,; m), \tag{18}$$

which is a sequence of the complex Fourier coefficients Z_m, for $m = -N/2, \cdots, +N/2$. The Fourier series which interpolates the z_j is

$$z(j) = {\sum_{m=-N/2}^{+N/2}}'' Z_m\, e^{+i2\pi mj/N}, \tag{19}$$

where the sum Σ'' is interpreted in a trapezoidal rule sense, with a value of 1/2 multiplying the end contributions at $\pm N/2$. For the case of integer j, this is the inverse discrete transform, denoted by the symbol D^{-1}:

$$z_j = D^{-1}(Z_m\,; j), \tag{20}$$

although in keeping with the approach of this paper we have not yet adopted integer values for the j in equation (19). It can be differentiated to give:

$$z_j' = \frac{i2\pi}{N} {\sum_{m=-N/2}^{+N/2}}'' mZ_m\, e^{+i2\pi mj/N} = \frac{i2\pi}{N} D^{-1}(mZ_m\,; j)\,. \tag{21}$$

In this way, if fast Fourier transform programs are available, the z_j' may be computed by taking the discrete Fourier transform of the points z_j, multiplying each coefficient by m and inverting, all of which can be done in $O(N \log N)$ operations.

7. SET-UP AND SOLUTION OF SYSTEM OF EQUATIONS

When the z_j' have been obtained, the coefficients $\Omega_{mj} = \alpha_{mj} + i\beta_{mj}$ can be calculated and used in expressions (17a) and (17b), one for each point at which an unknown exists. As the equations are nearly diagonally dominant, however, it should be possible to exploit the simple Gauss-Seidel iterative procedure, particularly for timestepping problems such as those for wave propagation, and in practice this was found to work very well indeed. The computational effort is $O(N^2)$ per iteration, and the happy result was found in the present work, that as all boundary points are interpreted as Lagrangian particles, and carry the geometry of the problem with them, then the coefficients are very slowly varying, and a forward extrapolation of previous results gave such an accurate initial estimate that only one iteration was usually necessary each time step to achieve an accuracy of seven figures.

Much programming detail can be avoided if the step of assembling into a matrix is bypassed. In this case, equations (17a) and (17b) may simply be rewritten: for points on the free surface

$$\psi_m = \frac{-d\phi(m)/dm - \sum_{j=0, j\neq m}^{N-1} \left(\alpha_{mj}(\phi_j - \phi_m) - \beta_{mj}\psi_j\right)}{\sum_{j=0, j\neq m}^{N-1} \beta_{mj}}, \tag{22a}$$

and for points on the sea bed

$$\phi_m = \frac{d\psi(m)/dm + \sum_{j=0, j\neq m}^{N-1} \left(\alpha_{mj}(\psi_j - \psi_m) + \beta_{mj}\phi_j\right)}{\sum_{j=0, j\neq m}^{N-1} \beta_{mj}}. \tag{22b}$$

In practice, a procedure of over-relaxation can be adopted to give faster convergence. It was found convenient in the present work where the coefficients changed slowly, not to store all the coefficients α_{mj} *etc.*, as this requires storage of $O(N^2)$, but to generate the coefficients necessary for each equation every time it had to be evaluated such that the storage was $0(N)$, and large numbers of points could be used. Overall, the implementation of the scheme in this iterative form was particularly simple and rapid.

8. RESULTS

The first test of the method was on the steady propagation of a wave. A test wave was used of height 25% of the mean depth and a length 20 times that of the depth. The initial conditions were computed using the accurate numerical method described in reference (7), and the wave was then placed in a computational "tank" twice its length, so that what was being simulated was the propagation of a solitary wave. Results are shown in Figure 2.

It seems that the method is indeed capable of accurately

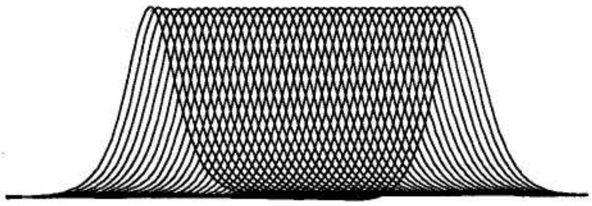

Figure 2. Results from the present method for the propagation of a steady wave of translation.

describing the propagation of a steady wave of translation with practically no distortion as it propagates, giving some confidence for its accuracy in other applications. There is a slight dispersive tail being generated as the wave propagates, although that may be due to the artifice of joining a steady wave up to an undisturbed section in a rectangular box. The computational resources required to obtain these results were demanding, however: a total of 224 boundary points were used: 104 on the free surface and 40 on each of the solid boundaries, left, right and bottom. A computational time step of 0.01 was used (dimensionless with respect to depth and gravitational acceleration). The figure shows the results from every 50th step, with a total of some 2000 steps, taking about an hour on a personal computer. It is an unfortunate characteristic of the present method at this stage of development, that as equally-spaced points were necessary it did seem to need quite large numbers of points on the sides, even though the water is shallow. That is because of the global or Fourier nature of the present method - even to describe the undisturbed rectangular tank to that accuracy, it would also take that number of points. The convenience of the Fourier method does seem to come at some cost, although plainly it is very accurate indeed.

What was surprising and disappointing to the author was the fact that if the wave were allowed to propagate until it slammed up against the wall on the right side, computations became unstable and no reliable results could be obtained. This may be an artefact of the computer program, as the high accuracy of the method should carry over even to the wave reflection problem.

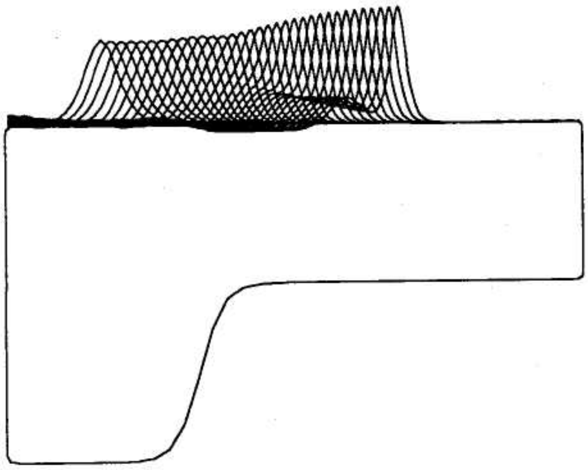

Figure 3. Results from the present method for the propagation of a wave over a shelf of depth 50%.

Figure 3 shows the results for a rather more interesting problem, where the same wave of Figure 2, of height 25% of the depth and a length 20 times the depth was allowed to propagate across a shelf, which shoaled to half the depth in a distance of roughly the horizontal length scale of the wave.

The results show some of the interesting phenomena associated with this nonlinear problem, although they are rather confused on the left side where the small waves generated by the interaction with the shelf have travelled to the left wall and been reflected there, obscuring the results. It is interesting that little seems to happen to the wave until it has travelled almost right across the shelf of about its own effective wavelength. Then, quite quickly the wave starts to grow in height, now travelling over water of constant shallower depth, and the large feature of a shelf develops behind the wave, which seems to be in the process of separating from the main wave and possibly becoming part of an oscillatory tail. The actual height of the wave achieved is about 35% greater than the original, rather more than the 19% which would be predicted by Green's Law, Massel (8), based on all variation being long. However, the main body of the wave now seems to be propagating without much change in this new depth.

Another run was made, whose results are shown in Figure 4, where the same wave was used, but where the sea bed came up from a dimensionless depth of 1 to 0.25, so that the effect on the wave should be so greater, and it was hoped the method might describe overturning and

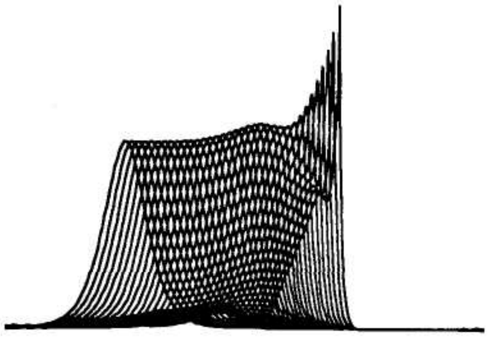

Figure 4. Results from the present method for the propagation of a wave over a shelf of depth 25%.

plunging of the wave. It can be seen that qualitatively, the results are similar to Figure 2. However, the results for the present computational method are disappointing, as it seems to have been unable, with the computational parameters used, to resolve the crest of the wave or to describe latter stages of its evolution. Another shelf was obtained, but in this case the wave crest continued to grow in height and sharpness as shown, but where there was insufficient computational resolution to describe the growth accurately or the probable overturning and plunging of the wave crest.

9. CONCLUSIONS

A numerical method has been developed for the numerical solution of Laplace's equation, which has been shown to have a number of desirable features and advantages over traditional methods for the accurate solution of potential problems. The method has some features which suggest that it might be a powerful tool in the numerical simulation of wave shoaling problems, as it handles irregular geometries easily, and has the potential to be considerably more accurate than other methods, and is computationally more robust. It has a feature which also suggests itself for unsteady wave propagation problems, that the equations are nearly diagonally dominant, and simple Gauss-Seidel iteration with over-relaxation could be used, which worked very efficiently, as the converged solution at one time provided an accurate initial solution for the next time step.

Some simple problems of wave propagation were solved, and the method was found to be powerful for some, and to provide interesting results, where a wave of moderate amplitude encounters a realistically varying seabed, the method was found to be accurate, possibly more so than other boundary integral methods. The problem of a shelf with a vertical face, or other sub-surface geometry would present no problems.

However for some problems it was found to be not as robust as had been hoped. For example, it was found that the precise and accurate point placement necessary for very high accuracy could not be guaranteed for time stepping problems, and the simplest equally-spaced method was used.

The method was found not to be able to handle the problem of wave reflection from a wall, although that may be a difficulty with the computer programming rather than the method.

Also, at the present stage of development and computer resources, it was not able to describe the overturning and plunging of a wave where the bottom shoaled dramatically.

As the method does have the ability to solve Laplace's equation to exceptionally high accuracy on a fixed domain, its most appropriate application might be to methods such as those of Leitao and Fernandes (4) which use such a domain with approximate boundary conditions.

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SESSION 2