A Numerical Cnoidal Theory for Steady Water Waves

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Summary: A method is presented for the numerical solution of the full nonlinear problem of waves propagating steadily over a flat bed. In recent years if high accuracy has been required the problem has been solved by methods involving Fourier approximation. For very long waves, such Fourier methods become inefficient. This paper introduces a similar numerical method, but where approximation is in terms of elliptic en functions rather than trigonometric functions, as suggested by conventional cnoidal theory. Results are presented which show that the method is accurate for waves longer than some eight times the water depth, and treats very long waves apparently without difficulty. As the theoretical highest waves are approached, the accuracy decreases to an approximate engineering accuracy. However this limit is unlikely to be reached in practice, and the method should provide an accurate and convenient method for all practically-possible long waves and, with some more development work, possibly short ones as well.

1. INTRODUCTION

A convenient wave model for many applications in coastal and ocean engineering is that of a periodic wave train propagating steadily without change of form in water of constant depth. There are three main approaches to solving this problem accurately. The first is to use a Stokes theory, which is based on Fourier representation of the wave, but where the coefficients are found by series expansions in terms of the ratio of wave height to length. To lowest order this yields linear wave theory. It has been shown ([1]) that results from these are of surprisingly high accuracy provided the waves are not too long relative to the depth. The second approach, more suited to shallow water, is to use a cnoidal theory, where the name comes from the Jacobian elliptic en functions which are used, and is based on an assumption that the wave motion is long relative to the depth,. The approach of conventional cnoidal theory is to express the expansions in terms of the ratio of wave height to water depth. Results for fluid velocity have been shown to be erratic, ([2]). In [1] it has, however, been shown that results from cnoidal theory are also of surprisingly high accuracy if, instead of being converted to series in terms of wave height to depth, the series are expressed in terms of shallowness, the ratio of water depth to wave length. However, for very high waves and for shorter waves, even the modified cnoidal theory becomes inaccurate.

For reliable and highly accurate solutions, numerical solution of the resulting nonlinear system of equations has become the preferred method of solution. To date, these methods have used Fourier approximation, but where the coefficients are found numerically instead of using perturbation expansions (see, for example, [3]). One disadvantage of these methods is that they are not so efficient for long waves, the Fourier approximation suffering from having to approximate both the short rapidly varying crest and the long trough where very little is changing, so that large numbers of terms and computational points are necessary, which can be very demanding, as matrix solution methods are necessary.

This paper describes a new theory which occupies an obvious gap in the different approaches described above: it solves the full nonlinear equations numerically, but uses cnoidal functions as the fundamental means of approximation, so that very long waves can be treated without any special methods. It is found that the method can be used for waves whose length is greater than eight times the water depth, and gives highly accurate results for all waves longer than this. For physically-realisable wave heights it is very accurate, but if the wave height is approaching that of the theoretical maximum, when the accuracy downgrades to approximate engineering accuracy. The method is still under development; it has not yet been extended to the case where the wave period instead of wavelength is specified.

2. THEORY

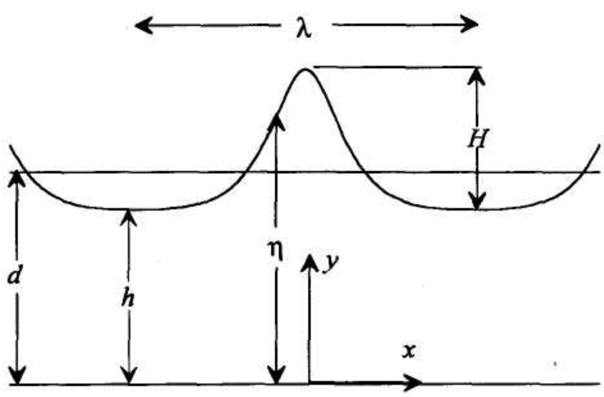


Figure 1: Wave, showing important dimensions

Figure 1 shows a typical steady wave, symmetric about the crest, with the sharp crest and long trough characteristic of steep and shallow waves. The (x,y) coordinate origin is on the bed under the crest and moves with the crest. In this frame all motion is steady. The physical dimensions shown are the mean depth d, the wave height H, the wavelength λ, the elevation of the free surface above the flat bed at any point, η, and the depth under the trough h.

It is assumed that the water is incompressible and the flow irrotational and two-dimensional, such that a stream function $\psi(x,y)$ exists and satisfies Laplace's equation throughout the flow:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0. \quad (1)$$

The boundary conditions are that the bottom is a streamline on which ψ is constant:

$$\psi(x,0)=0, \quad (2)$$

and that the free surface is also a streamline:

$$\psi(x,\eta(x))=-Q, \quad (3)$$

where Q is the volume flux per unit span underneath the wave train. The negative sign is because the flow relative to the wave is in the negative x direction, such that relative to the water the waves will propagate in the positive x direction. The remaining boundary condition comes from Bernoulli's equation: the condition that the pressure on the free surface is constant is expressed by

$$\frac{1}{2}\left(\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right)\Bigg|_{y=\eta}+g\eta=R, \quad (4)$$

in which g is gravitational acceleration, R is the Bernoulli constant, and we have used the result that the velocity (u,v) is given by

$$u=\frac{\partial\psi}{\partial y} \text{ and } v=-\frac{\partial\psi}{\partial x} . \quad (5)$$

We assume a Rayleigh expansion for ψ of the form:

$$\psi=-\sin y\frac{d}{dx}.f(x)=-y\frac{df}{dx}+\frac{y^3}{3!}\frac{d^3f}{dx^3}-\ldots, \quad (6)$$

as in [2]. This satisfies the field equation (1) and the bottom boundary condition (2) identically. The kinematic surface boundary condition (3) becomes

$$\sin\eta\frac{d}{dx}.f(x)=Q , \quad (7)$$

This equation is a nonlinear ordinary differential equation for the local fluid depth η and $f'(x)$, the local fluid velocity on the bed, in terms of the horizontal coordinate x.

It is more convenient to write the equations in terms of dimensionless variables. We introduce the scaled horizontal variable $\theta=\alpha x/h$, where α is a stretching parameter. Writing $\eta_*=\eta/h$, equation (7) can be written

$$\sin\eta_*\alpha\frac{d}{d\theta}.\frac{f}{Q}-1=0. \quad (8)$$

Introducing $f_*'(\theta)=h/Q\, df/dx$ and the series expansion of the sin operator, the equation becomes:

$$\eta_* f_*'(\theta)+\sum_{i=1}^{\infty}\left((-1)^i\frac{\eta_*^{2i+1}}{(2i+1)!}\left(\alpha\frac{d}{d\theta}\right)^{2i}f_*'(\theta)\right)-1=0. \quad (9)$$

We can write the dynamic surface boundary condition (4) as:

$$\frac{1}{2}\left(\left(\frac{u_s h}{Q}\right)^2+\left(\frac{v_s h}{Q}\right)^2\right)+\frac{gh^3}{Q^2}\frac{\eta}{h}=\frac{Rh^2}{Q^2}, \quad (10)$$

after multiplying by $(h/Q)^2$ to make it non-dimensional. The velocity components are obtained by differentiating equation (5) to give, in dimensionless terms:

$$u_{*s}=\frac{u_s h}{Q}=-\cos\alpha\eta_*\frac{d}{d\theta}.f_*', \quad (11a)$$

$$v_{*s}=\frac{v_s h}{Q}=\sin\alpha\eta_*\frac{d}{d\theta}.f_*'. \quad (11b)$$

Writing these as expansions we have

$$u_{*s}=-f_*'-\sum_{i=1}^{\infty}\left((-1)^i\frac{(2i+1)\eta_*^{2i}}{(2i+1)!}\left(\alpha\frac{d}{d\theta}\right)^{2i}f_*'\right), \quad (12a)$$

$$v_{*s}=\alpha\sum_{i=1}^{\infty}\left((-1)^i\frac{\eta_*^{2i+1}}{(2i+1)!}\left(\alpha\frac{d}{d\theta}\right)^{2i}\frac{df_*'}{d\theta}\right), \quad (12b)$$

in which the factor of $2i+1$ has been retained in the denominator of (12a) because the factorial term in the denominator will be absorbed into subsequent coefficients. The dynamic boundary condition (10) can be written in terms of these quantities as

$$\frac{1}{2}\left(u_{*s}^2+v_{*s}^2\right)+g_*\eta_*=R_*, \quad (13)$$

where $g_*=gh^3/Q^2$ and $R_*=Rh^2/Q^2$.

We use a spectral approach, in which all functions of x are approximated by polynomials of degree N in terms of the square of the Jacobian elliptic function $\mathrm{cn}^2(\theta|m)$ for the surface elevation and bottom velocity of the form suggested by conventional cnoidal theory:

$$\eta_*=1+\sum_{j=1}^{N}Y_j\,\mathrm{cn}^{2j}(\theta|m), \quad (14)$$

$$f_*'=F_0+\sum_{j=1}^{N}F_j\,\mathrm{cn}^{2j}(\theta|m), \quad (15)$$

where the Y_j and F_j are numerical coefficients for a particular wave. Conventional cnoidal theory expresses the coefficients as expansions in terms of the parameter α which is related to the shallowness (depth/wavelength)^2, and produces a hierarchy of equations and solutions based on series expansions in terms of α, which is required to be small. In this work we attempt to solve the equations by making no expansions in terms of physical quantities, and we seek numerical solutions, in a manner similar to that in which Fourier approximation methods relate to Stokes theory for steady water waves.

On substituting equations (14) and (15) into equations (9) and (13) we have the general problem of evaluating an arbitrary derivative of an arbitrary even power of $\mathrm{cn}(\theta|m)$:

$$\left(\frac{d}{d\theta}\right)^{2i}\mathrm{cn}^{2j}(\theta|m), \quad (16)$$

for any integer i and j. Incorporating the factorial in the denominator of equation (12), this can be expressed as the double sum

$$\frac{1}{(2i+1)!}\left(\frac{d}{d\theta}\right)^{2i}\mathrm{cn}^{2j}(\theta|m)=\sum_{p=0}^{i+j}\sum_{q=0}^{i}A_{ijpq}\,\mathrm{cn}^{2p}(\theta|m)\,m^q, \quad (17)$$

where the A_{ijpq} are purely numerical constants which only have to be computed once at the beginning of calculations. The algorithm for this is presented in the Appendix.

Substituting into equation (9):

$$\eta_* f_*' + \sum_{i=1}^{N-1}\left[(-1)^i \eta_*^{2i+1}\sum_{j=1}^{N-i} F_j \sum_{p=0}^{i+j} B_{ijp}\ \mathrm{cn}^{2p}(\theta|m)\right] - 1 = 0, \tag{18}$$

where we have replaced the sum which occurs throughout the work by the symbol

$$B_{ijp} = \sum_{q=0}^{i} A_{ijpq}\, m^q, \tag{19}$$

where the outer summation over i has only been taken as far as N-1 and the inner summation over j has only been taken as far as N-i, because from conventional cnoidal theory this would give a consistent accuracy to order α^{2N}.

Substituting equation (17) into equation (12) we have:

$$u_{*s} = -f_*' - \sum_{i=1}^{N-1}\left[(-1)^i (2i+1)\eta_*^{2i}\sum_{j=1}^{N-i} F_j \sum_{p=0}^{i+j} B_{ijpq}\ \mathrm{cn}^{2p}(\theta|m)\right] \tag{20a}$$

$$v_{*s} = \alpha \sum_{i=1}^{N-1}\left[(-1)^i \eta_*^{2i+1}\sum_{j=1}^{N-i} F_j \sum_{p=0}^{i+j} B_{ijpq}\frac{d}{d\theta}(\mathrm{cn}^{2p}(\theta|m))\right]. \tag{20b}$$

It can be shown that

$$\frac{d}{d\theta}(\mathrm{cn}^{2p}(\theta|m)) = -2p\,\mathrm{cn}^{2p-1}(\theta|m)\,\mathrm{sn}(\theta|m)\,\mathrm{dn}(\theta|m)\ , \tag{21}$$

for $p > 0$, such that

$$\frac{v_s h}{Q} = \alpha\eta\frac{df_*'}{d\theta} - 2\alpha\,\mathrm{cn}(\theta|m)\,\mathrm{sn}(\theta|m)\,\mathrm{dn}(\theta|m) \times$$
$$\sum_{i=1}^{N-1}\left[(-1)^i \eta^{2i+1}\sum_{j=1}^{N-i} F_j \sum_{p=1}^{i+j} p\,B_{ijpq}\ \mathrm{cn}^{2p-2}(\theta|m)\right], \tag{22}$$

in which

$$\frac{df_*'}{d\theta} = -2\,\mathrm{sn}(\theta|m)\,\mathrm{dn}(\theta|m)\sum_{j=1}^{N} jF_j\ \mathrm{cn}^{2j-1}(\theta|m). \tag{23}$$

When this quantity is squared in the Bernoulli equation (10), the relationships

$$\mathrm{sn}^2(\theta|m) = 1 - \mathrm{cn}^2(\theta|m)$$

and

$$\mathrm{dn}^2(\theta|m) = 1 - m + m\,\mathrm{cn}^2(\theta|m) \tag{24}$$

may be used, so that only the function cn need be used.

3. SYSTEM OF EQUATIONS

The free surface boundary conditions include the following unknowns: α, m, g_*, R_*, a total of N values of the Y_j for $i = 1\ldots N$, and $N+1$ values of the F_j for $i = 0\ldots N$, making a total of $2N+5$ unknowns. For the boundary points at which both boundary conditions are to be satisfied we choose M+1 points equally spaced in the vertical such that:

$$\mathrm{cn}^2\left(\alpha\frac{x_i}{h}|m\right) = 1 - i/M \ \text{ for } i = 0\ldots M, \tag{25}$$

where $i = 0$ corresponds to the crest and $i = M$ to the trough. This has the effect of clustering points near the wave crest, where variation is more rapid and the conditions at each point will be relatively different from each other. If we had spaced uniformly in the horizontal, in the long trough where conditions vary little the equations obtained would be similar to each other and the system would be poorly conditioned. We now have a total of $2M+2$ equations but so far, none of the overall wave parameters have been introduced. It is known that the steady wave problem is uniquely defined by two dimensionless quantities: the wavelength λ/d and the wave height H/d. In many practical problems the wave period is known, but this developmental paper will restrict consideration to those where the dimensionless wavelength λ/d is known. It can be shown ([2]) that λ/d is related to α by the expression which we term the Wavelength Equation:

$$\alpha\frac{\lambda}{d}\frac{d}{h} - 2K(m) = 0, \tag{26}$$

where $K(m)$ is the complete elliptic integral of the first kind, and where the equation has introduced another unknown d/h, the ratio of mean to trough depth.

The equation for this ratio is obtained by taking the mean of equation (14) over one wavelength or half a wavelength from crest to trough:

$$\frac{d}{h} = 1 + \sum_{j=1}^{N} Y_j\ \overline{\mathrm{cn}^{2j}(\theta|m)}. \tag{27}$$

The mean value of the power of the cn function over half a wavelength can be computed from the recurrence relation (see, for example, [2]), where if

$$I(p) = \frac{1}{K}\int_0^K (m\,\mathrm{cn}^2(\theta|m))^p\, d\theta\ ,$$

then $I(0) = 1$ and $I(1) = -1 + m + E(m)/K(m)$, where $K(m)$ and $E(m)$ are complete elliptic integrals of the first and second kinds, and

$$I(p+2) = \frac{2p+2}{2p+3}(2m-1)I(p+1) + \frac{2p+1}{2p+3}(m-m^2)I(p), \tag{28}$$

for $p = 0, 1, \ldots$. Then,

$$\overline{\mathrm{cn}^{2j}\left(\alpha\frac{x}{h}|m\right)} = \frac{I(j)}{m^j} \ \text{ for all } j = 1\ldots N, \tag{29}$$

and equation (27) can be written

$$1 + \sum_{j=1}^{N} Y_j\frac{I(j)}{m^j} - \frac{d}{h} = 0, \tag{30}$$

thereby providing one more equation, the Mean Depth Equation.

Finally, another equation which can be used is that for the wave height:

$$\frac{H}{h} = \frac{\eta_0}{h} - \frac{\eta_M}{h}, \tag{31}$$

which, on substitution of equation (14) at $x = x_0 = 0$ where $\mathrm{cn}(0|m) = 1$ and, because $\mathrm{cn}(\alpha x_M|m) = 0$ from equation (25), gives

$$\frac{H}{d}\frac{d}{h} - \sum_{j=1}^{N} Y_j = 0, \tag{32}$$

the Wave Height Equation.

The system of nonlinear equations which we have to solve is listed in Table 1. We write the system of equations as

$$\mathbf{e}(\mathbf{z}) = \{e_i(\mathbf{z}),\ i = 1\ldots 2M+5\} = \mathbf{0}, \tag{33}$$

where e_i is the equation with reference number i as given in the table, and where

$$\mathbf{z} = \{z_j,\ j = 1\ldots 2M+6\}, \tag{34}$$

where z_j is the unknown with reference number j as given in the list of unknowns in Table 2.

Equation	Number of Equations	Reference Number
Wavelength Equation (26)	1	1
Wave Height Equation (32)	1	2
Mean Depth Equation (30)	1	3
Kinematic free surface	M+1	4..M+4
Dynamic free surface	M+1	M+4..2M+5
Total	2M+5	

Table 1: System of nonlinear equations

The variables which are used are set out in Table 2. Whereas the parameter m has been used in the above development, it has the unpleasant property that it has a singularity in the limit as $m \to 1$, which corresponds to the long wave limit, and as we will be using gradient methods to solve the nonlinear equations this might make solution more difficult. It is more convenient to use the ratio of the complete elliptic integrals as the actual unknown, which we choose to be the first:

$$z_1 = \frac{K(m)}{K(1-m)}. \tag{35}$$

The remaining variables and their reference numbers are shown in Table 2, and will be referred to as z_j, where j is the reference number in the right column.

Variable	Number	Reference Number
$K(m)/K(1-m)$	1	1
α	1	2
d/h	1	3
$g_* = gh^3/Q^2$	1	4
$R_* = Rh^2/Q^2$	1	5
$F_j,\ j=0\ldots N$	N+1	6..N+6
$Y_j,\ j=1\ldots N$	N	N+7..2N+6
Total	2N+6	

Table 2: List of variables

4. NUMERICAL SOLUTION

The solution of the system of nonlinear equations follows that in [3], using Newton's method in a number of dimensions. We denote the system to be solved, equation (33), as

$$\mathbf{e}(\mathbf{z}) = \mathbf{0}. \tag{36}$$

To solve this nonlinear system an iterative procedure will be used. If $\mathbf{z}^{(n)}$ is the approximate solution vector after iteration n, then evaluating the left side of equation (36) will give, not zero, but $E_i^{(n)}$:

$$e_i(\mathbf{z}^{(n)}) = E_i^{(n)} \quad \text{for all } i = 1\ldots 2M+5. \tag{37}$$

If we were to compute an estimate of the errors at the next iteration we could write the system of equations as a multi-dimensional Taylor expansion:

$$E_i^{(n+1)} = E_i^{(n)} + \sum_{j=1}^{2N+6} \frac{\partial E_i^{(n)}}{\partial z_j^{(n)}} \left(z_j^{(n+1)} - z_j^{(n)} \right) + \ldots, \tag{38}$$

and, for computational purposes we require that this expansion be truncated at the order shown and assume that this would give $E_i^{(n+1)} = 0$ for all the i, so that the equation can be written as a matrix equation which we can solve for the updated solution vector $\mathbf{z}^{(n+1)}$:

$$\left[\frac{\partial E_i^{(n)}}{\partial z_j^{(n)}} \right] (\mathbf{z}^{(n+1)} - \mathbf{z}^{(n)}) = -\mathbf{E}^{(n)}, \tag{39}$$

in which the matrix element at (i,j) is the derivative of equation i with respect to unknown j as shown. It would be possible to obtain the derivatives analytically, however, particularly in the surface boundary condition equations the unknowns appear embedded very deeply. It is far simpler to obtain the derivatives by numerical differentiation: adding an arbitrary small quantity δz_j to variable z_j, evaluating all the equations, and taking the appropriate differences, as done in [3].

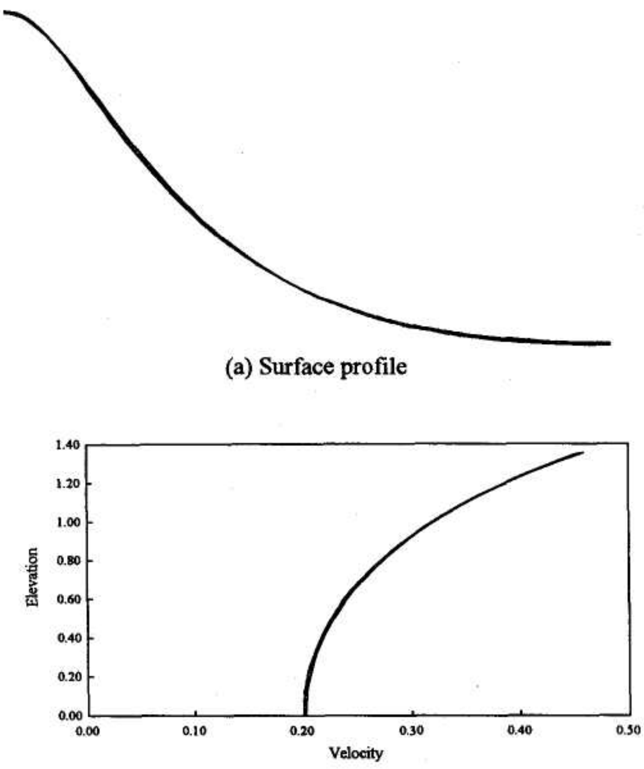
As the number of equations and variables can never be the same (2M+5 can never equal 2N+6 for integer M and N), we have to solve this equation as a generalised inverse problem. Fortunately this can be done very conveniently by the Singular Value Decomposition method (for example Press *et al.*,[4], #2.6) so that if there are more equations than unknowns, $M > N$, the method obtains the least squares solution to the overdetermined system of equations. In practice this was found to give a certain rugged robustness to the method, despite the equations being rather poorly conditioned.

5. INITIAL CONDITIONS

In the computations reported in this paper the simplified fifth order cnoidal theory presented in [1] was used. The first step is to compute an approximate value of m and hence z_1 using the analytical expression for wavelength in terms of m given as equation (19) in [1], combined with the bisection method of finding the root of a single transcendental equation. After that the rest of the fifth order expressions presented in [1] can be used. It was found, in performing computations for this paper, that providing only first order results as the approximate initial solution was not enough for waves which are high or not so long, when the first order theory becomes less accurate.

6. RESULTS

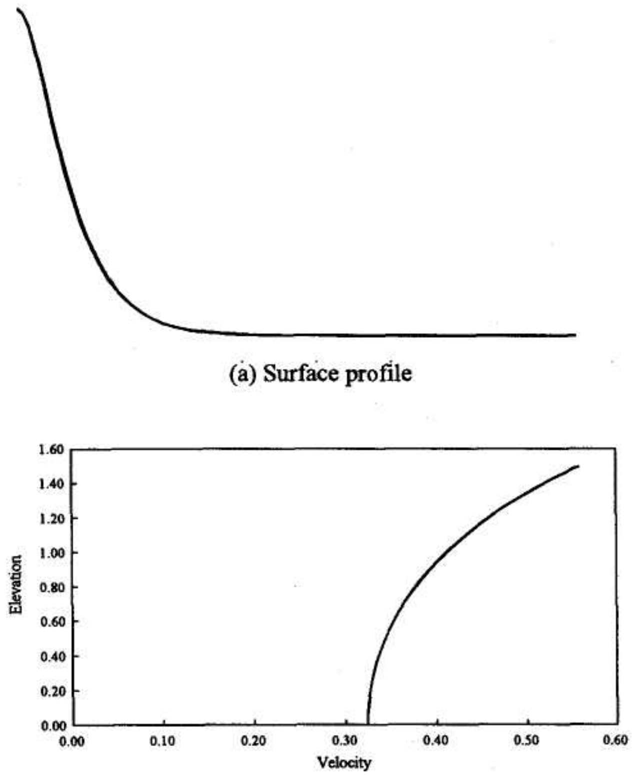
Figure 2 shows the solution obtained for a high wave of intermediate length, when conventional cnoidal theory is considered not valid. In (a), the surface profiles are shown, and in (b), the fluid velocity profiles under the crest are shown. Two curves are plotted on each, results from the present method and those from the Fourier approximation method, which should be highly accurate in this relatively short wave limit. It can be seen that the results are almost indistinguishable at the scale of plotting. Whereas conventional cnoidal theory should not be valid in this shorter wave limit, as it depends on the waves being long for its accuracy, there is nothing in the present numerical method which



(a) Surface profile

(b) Velocity profile under crest

Figure 2: Results for a high intermediate wave of $H/d = 0.5$ and $\lambda/d = 8$ solved by the Fourier approximation method with 25 terms and the present theory with $N = 10$.



(a) Surface profile

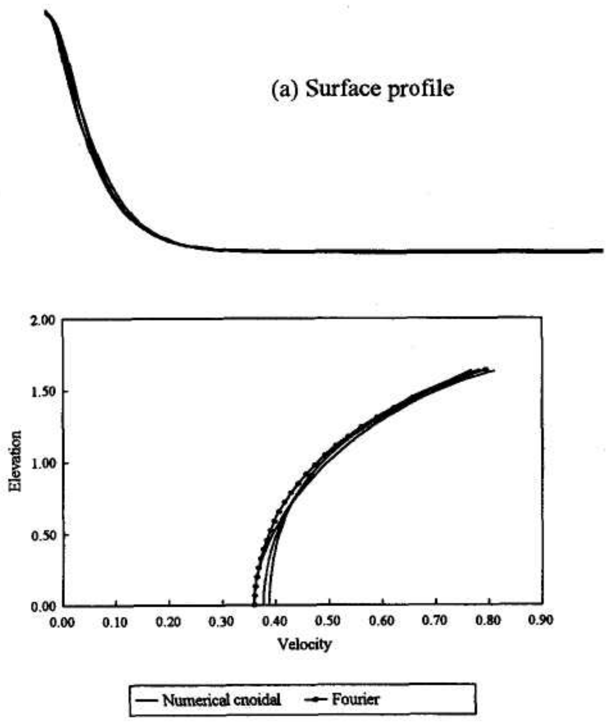
(b) Velocity profile under crest

Figure 3: Results for a high long wave of $H/d = 0.55$ and $\lambda/d = 30$ solved by the Fourier approximation method with 30 terms and the present theory with $N = 9$. Results are almost coincident.

necessarily limits its accuracy to long waves. In the preparation of this paper, for the initial conditions only cnoidal theory was used, and it was not accurate enough for waves shorter than this example. If Stokes theory could be modified to provide the initial conditions, there is no reason why the present method could not be used for considerably shorter waves.

Figure 3 shows the results for a long wave, of *Hid* = 0.55 (the maximum practical height, [5]), and *X/d* = 30. This wave is sufficiently long that the Fourier method is beginning to be tested considerably, yet it is capable of giving results provided sufficient numbers of Fourier terms are taken and the user waits long enough. It can be seen that the present numerical cnoidal theory is also capable of high accuracy, as demonstrated by the close agreement between the two very different theories. It used much smaller computing resources, typically involving the solution of systems of 25 equations compared with 70 equations for the Fourier method.

Figure 4 shows the behaviour of the numerical cnoidal method for very high waves. The Fourier method was at the limits of its powers for this problem, requiring 14 steps in height for the method to converge. Results from the numerical cnoidal theory for N= 7, 8 and 9 show certain irregularities, and the method was having trouble converging for this wave, which is closer to the theoretical maximum for waves of that length. Although the method shows difficulty with convergence, it does yield results of engineering accuracy.



(a) Surface profile

(b) Velocity profile under crest

Figure 4: Results for a very high long wave of $H/d = 0.7$ and $\lambda/d = 30$, showing results from the Fourier method with 30 terms and various applications of the present theory with $N = 7$, 8 and 9.

There is evidence that no long wave in shallow water can exist at this height, and that a maximum of $H/d = 0.55$ is more likely [5], and that the accuracy of Figure 3 is more likely to be that which obtains in practice.

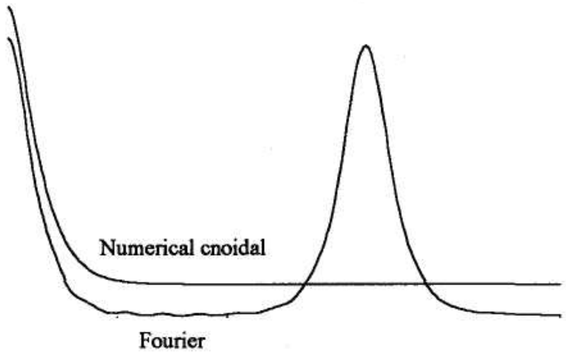


Figure 5: Results for a high very long wave of $H/d = 0.55$ and $\lambda/d = 50$, showing the present theory with $N = 10$ and the Fourier method with 30 terms.

Figure 5 shows some of the difficulties associated with the Fourier method for very long waves. It is well-known that for long waves the Fourier method may converge to a wave of 1/3 the wavelength [6], but that this can be remedied by solving for lower waves of the same length and stepping upwards in height. In the figure, some 10 such steps were taken, using 30 terms in the Fourier series, a rather lengthy computation, but the method still failed to converge to the correct solution. The numerical cnoidal method, however, converged to give an accurate solution to this rather demanding problem.

7. CONCLUSIONS

A method has been presented for the numerical solution of the full nonlinear problem of waves propagating steadily over a flat bed, where approximation is in terms of elliptic functions so that problems of very long waves can be solved. Results are presented which show that the method is accurate for waves longer than some eight times the water depth, and treats very long waves apparently without difficulty. As the highest waves are approached, however, the accuracy decreases to an approximate engineering accuracy. The method is still under development, and it is hoped that it will be able to solve even deep water problems provided reasonably accurate initial estimates can be provided.

8. REFERENCES

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9. APPENDIX: CALCULATION OF TABLE OF COEFFICIENTS

Set all $A[i,j,p,q] = 0$

Remark: Table of second derivatives

for p from 1 to N do

$A_{1,p,p-1,0} := p(-1+2p)/3$
$A_{1,p,p-1,1} := p(1-2p)/3$
$A_{1,p,p,0} := p(-2p)/3$
$A_{1,p,p,1} := p(4p)/3$
$A_{1,p,p+1,1} := p(-1-2p)/3$

Remark: Calculate higher derivatives recursively:

for i from 2 to N do
for p from 1 to N-i+1 do
for q from 1 to p+i-1 do
for j from 0 to i-1 do
for J from 0 to 1 do
for k from 0 to p+i do

$$A_{i,p,k,j+J} := A_{i,p,k,j+J} + \frac{3}{i(2i+1)} A_{i-1,p,q,j} A_{1,q,k,J}$$