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Long wave equations for waterways curved in plan

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INTRODUCTION

The St Venant equations hold a central place in the theory of the propagation of waves and floods in open channels. Their one-dimensional simplicity is in keeping with the approximately known flow and geometry of the problem, however the effects of curvature of the stream have generally been neglected. The present paper is an attempt to incorporate the effects of channel curvature on the propagation of floods and long waves, retaining the relative simplicity of the one-dimensional equations. The effects of stream curvature are expressed only by the offset of the centroid of the cross section and the middle of the stream relative to the local radius of curvature. The essential structure of the equations is the same as the traditional form, and the usual methods may be modified relatively simply to solve them. An expression is given for the speed of long waves as modified by curvature effects in arbitrary channels. It is seen that in real rivers floods travel faster due to the effects of curvature than they would in the same channel straightened out fictitiously for computational purposes, which may have important practical consequences.

CURVILINEAR COORDINATE SYSTEM

Consider a waterway of arbitrary cross-section, which meanders such that it is curved in plan as shown in Figure 1. Let there be some arbitrary curve along the river, where the distance along the curve is denoted by s, and at any point on this curve let there be a local orthogonal curvilinear co-ordinate system (s, n, z), where n is horizontal and transverse to s, and z is vertically upwards. The local radius of curvature of the reference axis is r, such that in an elemental increment ds it turns through an angle $d\theta$, then $ds = r\,d\theta$. An elemental unit of volume at (s, n, z) is shown in plan; its component lengths are $(r - n)/r\,ds$ (by proportionality), dn, and dz, but it is more convenient to introduce the curvature $\kappa = 1/r$, so that the elemental dimensions are $(1 - \kappa n)ds$, dn, and dz.

The approach in this work parallels that of Fenton (1995) for straight channels. Consider a control volume as shown in plan in the figure, consisting of a slice of the channel bounded by the two vertical planes perpendicular to the local s axis at s and s+ds. The control volume made up of this elemental section is continued into the air such that the lateral boundaries are the river banks, the upper boundary being arbitrary but never intersected by the water. The control volume thus contains liquid of constant density ρ in the lower part and air of

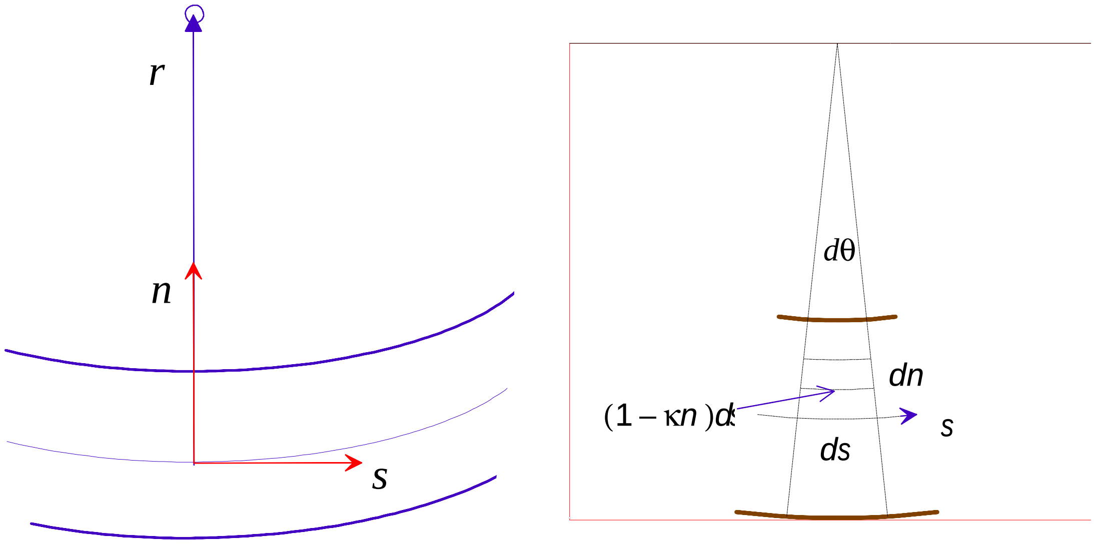

Figure 1. Orthogonal curvilinear coordinate system, physical dimensions, and control volume

negligible density in the upper part, whose motion does not contribute to mass and momentum exchange. This choice of control volume is important. If the upper surface of the liquid had been chosen as the upper boundary, complicated manipulations including the kinematic boundary condition on the free surface would have to be performed.

Although the choice of position of the s axis is arbitrary, it will usually be reasonable to choose it so that it follows the course of the river, possibly chosen to be the midpoint of the surface of the river at a particular stage, as taken from cross-sections or aerial photographs, or chosen to be the path followed by the deepest part or talweg.

MASS CONSERVATION EQUATION

The Mass Conservation equation in integral form [Streeter and Wylie, 1981, #3.3] is:

$$\frac{\partial}{\partial t}\int_{CV}\rho\, dV+\int_{CS}\rho\,\mathbf{u}\,.\,d\mathbf{S}=0, \qquad (1)$$

where t is time, $\mathbf{u}$ is the velocity vector, and $d\mathbf{S}$ is a vector representing an area element of the control surface, with direction normal to and directed outwards from the control surface. It can be shown that the first term can be written

$$\rho\, ds\int_{n_R}^{n_L}(1-\kappa n)\frac{\partial\eta}{\partial t}\,dn, \qquad (2)$$

where $z=\eta(s,n,t)$ is the elevation of the free surface, and $n=n_L(s,t)$ and $n=n_R(s,t)$ are the coordinates of the water's edge at the left and right banks.

Considering the second term in equation (1), it is easily shown that the net mass leaving the control volume across the two vertical faces is $\rho\,\partial Q/\partial s\; ds$. If there is inflow from rainfall, groundwater or tributaries entering the channel at a volume rate of q per unit length, the extra mass leaving the control volume is $-\rho q\, ds$. Combining the individual contributions to equation (1), and as surface elevation may vary across the section, we use the cross-sectional area of the flow A instead, giving the Mass Conservation Equation:

$$(1-\kappa n_m)\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial s}=q+O\Big((\kappa n_L)^2,(\kappa n_R)^2\Big), \qquad (3)$$

where n_m is the transverse distance of the midpoint of the river surface from the streamwise co-ordinate. For a straight river, $\kappa=0$, and the usual expression is obtained. The terms which we have ignored, shown on the right side, are of the order of the square of the curvature times the distance from the reference axis to the bank. For the theory to be accurate the curvature and radius of curvature should then satisfy $(\kappa n_L)^2=(n_L/r)^2 << 1$ and $(\kappa n_R)^2=(n_R/r)^2 << 1$, such that the theory will be most accurate for streams where the width of the river is small compared with the radius of curvature. From now the order of accuracy will not be shown explicitly. Throughout this work, all mathematical operations will performed to give expansions in κn to first order. This level of approximation is similar to the usual one for the conventional formulation of the St Venant equations for channels presumed to be straight, where the approximation made is$(\text{depth / disturbance length})^2 << 1$, which will also later be made in this work.

MOMENTUM CONSERVATION EQUATION

Now consider the integral form of the Momentum Conservation Equation [Streeter and Wylie, 1981, #3.3] :

$$\frac{\partial}{\partial t}\int_{CV}\rho\,\mathbf{u}\,dV+\int_{CS}\rho\,\mathbf{u}\,\mathbf{u}\,.\,d\mathbf{S}=\mathbf{F}, \qquad (4)$$

where $\mathbf{F}$ is the force acting on the fluid in the control volume, including both surface and body forces. The contribution of pressure to the surface forces can be written as $-\int_{CS} p\, d\mathbf{S}$, where p is the pressure, the negative sign showing how the local force acts in the direction opposite to the outward normal. This term, in its form shown here, is difficult to evaluate for non-prismatic and/or curved channels, as the pressure and the non-constant unit vector have to be integrated over all the submerged faces of the control surface. A considerably simpler derivation is obtained if the term is evaluated using Gauss' Divergence Theorem [Milne-Thomson, 1968, #2.61], which states that the net pressure force on a closed surface equals the volume integral of the pressure gradient throughout the region enclosed by that surface. It is much easier to evaluate the volume integral of the pressure gradient than the surface area integral.

Combining the individual contributions, making the hydrostatic approximation for the pressure, and collecting all terms gives the momentum equation expressed in terms of integrals over the cross-sectional area:

$$\frac{\partial}{\partial t}\int_A(1-\kappa n)\,u\;dA+\frac{\partial}{\partial s}\int_A u^2\,dA+g\int_A\frac{\partial\eta}{\partial s}\,dA+g\,S_f\,A(1-\kappa\bar{n})=qu_q\,, \qquad (5)$$

in which g is gravitational acceleration, S_f is the friction slope, $\bar{n}$ is the transverse distance of the centroid of the cross-section from the streamwise coordinate, and u_q is the velocity of inflow before mixing.

Now the three integrals in this expression will be approximated for practical implementation. It can be shown, to within the accuracy of the derivation of the Mass Conservation Equation, where the effects of curvature are included to first order, that the contribution of the time derivative term is

$$(1-\kappa\bar{n})\frac{\partial Q}{\partial t}+\frac{Q}{A}(\kappa\bar{n}-\kappa n_m)\frac{\partial A}{\partial t}. \tag{6}$$

The next integral can be written

$$\int_A u^2 dA = \frac{Q^2}{A} \quad + \text{Neglected terms in second and higher derivatives of velocity.}$$

This is a surprising result, that the approximation is exact for linear vertical and horizontal shear flows. Traditionally it has been believed that setting the integral equal to Q^2/A is exact only if flow velocity is constant over the section. The term becomes, after differentiation:

$$\frac{\partial}{\partial s}\int_A u^2 dA \approx \frac{\partial}{\partial s}\left(\frac{Q^2}{A}\right) = 2\frac{Q}{A}\frac{\partial Q}{\partial s}-\frac{Q^2}{A^2}\frac{\partial A}{\partial s}. \tag{7}$$

The remaining term, $g\int_A \partial\eta/\partial s\, dA$ is a rather complicated expression to obtain, and the details will not be shown here. Expressing it in terms of the integrated quantities and collecting the other integral contributions to equation (5), equations (6) and (7), gives the Momentum Equation:

$$\begin{aligned}&(1-\kappa\bar{n})\frac{\partial Q}{\partial t}+\frac{Q(\kappa\bar{n}-\kappa n_m)}{A}\frac{\partial A}{\partial t}+\frac{2Q}{A}(1+\kappa n_m-\kappa\bar{n})\frac{\partial Q}{\partial s}\\ &\qquad+\left(\frac{gA}{B}-\frac{Q^2}{A^2}(1+2\kappa(n_m-\bar{n}))\right)\frac{\partial A}{\partial s}+\frac{Q^2\kappa'(n_m-\bar{n})}{A}-gA\bar{S}+gAS_f(1-\kappa\bar{n})-qu_q=0,\end{aligned}$$

where $\bar{S}$ is the mean downstream slope of the channel, and $\kappa' = d\kappa/ds$. It is more convenient to have just one time derivative in each equation, and so eliminating $\partial A/\partial t$ using the Mass Conservation Equation (3), gives:

$$\begin{aligned}&(1-\kappa\bar{n})\frac{\partial Q}{\partial t}+\frac{Q}{A}(2+3(\kappa n_m-\kappa\bar{n}))\frac{\partial Q}{\partial s}+\left(\frac{gA}{B}-\frac{Q^2}{A^2}(1+2(\kappa n_m-\kappa\bar{n}))\right)\frac{\partial A}{\partial s}\\ &\qquad=\frac{Q^2(\kappa'\bar{n}-\kappa' n_m)}{A}+gA\bar{S}-gAS_f(1-\kappa\bar{n})+q\left(u_q+\frac{Q}{A}(\kappa n_m-\kappa\bar{n})\right),\end{aligned} \tag{8}$$

where partial derivative terms have been retained on the left side while forcing terms have been taken to the right. As the friction slope S_f and inflow q are only approximately known, there is room for some simplification by dropping the curvature components in those terms.

DISCUSSION AND CONCLUSIONS

Equations (3) and (8) are the long wave equations which govern the propagation of floods and long waves in curved waterways, a pair of partial differential equations, which, provided boundary and initial conditions are specified, may be solved numerically. Importantly, the structure of the equations is the same as that of the conventional St Venant equations. The effects of curvature appear in them simply by the presence of the terms κn_m and $\kappa\bar{n}$ (and similar terms involving the derivative κ') in the coefficients and in the forcing terms on the right. These terms are, respectively, the ratio of the offset of the midpoint of the surface to the radius of curvature, and a similar ratio involving the offset of the centroid of the water cross section.

The two partial differential equations can be written as ordinary differential equations in a characteristic formulation, from which it can be deduced that c, the speed of propagation of disturbances relative to still water, is given by

$$c = \sqrt{\frac{gA}{B}}\left(1 + \frac{1}{2}\kappa n_m + \frac{1}{2}\kappa\bar{n}\right). \tag{9}$$

It is well known that C, the speed of long waves in a straight channel, is given by $C = \sqrt{gA/B}$, where A/B is capable of simple interpretation as the mean depth. It is perhaps surprising that the modification for a curved channel is so simple. However, it is not so simple to put a physical interpretation on the length scale $\sqrt{A/B} \times \left(1 + \frac{1}{2}\kappa n_m + \frac{1}{2}\kappa\bar{n}\right)$ in this equation. Unfortunately, the simple interpretation that the square of the wave speed is given by $g \times \text{Mean Depth}$ does not seem to hold in this case where the channel is curved in plan.

For real rivers, which tend to be shallower on the inside of bends and steeper on the outside, the location of the streamwise axis will usually be determined by normal flows in the lower part of the bed. As the water level rises in a flood, the surface spreads out more over the shallower inside bank causing both n_m and $\bar{n}$ to increase. This remains true for both positive and negative curvature: for typical rivers both n values move in the direction of the centre of curvature and both κn_m and $\kappa\bar{n}$ are positive, whether the river curves to left or right.

This has important implications for flood prediction, as the speed of propagation of floods and waves in such real rivers is then greater than that given by the straight channel approximation. Even if all the computational details of the present method were not to be included in operational models, there should be at least some allowance made for this effect.

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