Proc. 10th Congress, Asia-Pacific Division
International Association for Hydraulic Research
Langkawi, Malaysia, 1996

An examination of the approximations in river and channel hydraulics

John D. Fenton

Department of Mechanical Engineering, Monash University
Clayton, Victoria, Australia 3168

Introduction

This paper obtains and examines the principal approximations for describing the movement of long waves and floods in waterways. Traditionally the St Venant long wave equations have occupied a central position in this area. They have been studied at great length and form the basis of many numerical models. In this paper a new higher-order approximation is derived which does not make the traditional hydrostatic approximation, which may be useful in practice. These so-called Boussinesq equations, named after similar long wave equations in coastal engineering, contain a third derivative term. If this term is deleted a form of the St Venant equations is obtained which uses cross-sectional area and discharge as variables, rather than surface elevation and velocity. Subsequently both the Boussinesq and St Venant equations are linearised about a uniform steady state, and it is shown that the behaviour of solutions of these equations is rather more complicated than has generally been realised. As expected, the solutions show wavelike behaviour, but they also show diffusive and dispersive behaviour, where the decay and propagation behaviour of the solutions show dependence on wavelength. While dispersion is to be expected for the Boussinesq form, it has not been associated with the St Venant equations. A novel inference from this is that the velocity of propagation of waves in channels with friction is *not* the widely-presumed long wave speed.

Exact equations

By considering a control volume based on a slice of an open channel of elemental length, (for example, Cunge *et al.* (1980, p.14)) it can relatively easily be shown that the requirement that mass be conserved is given by the *mass conservation equation*:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q, \tag{1}$$

where A is the cross-sectional area of the flow, Q is the discharge in the channel, q is the volume rate per unit length at which inflow from rain groundwater or tributaries enters the channel, x is the distance along the channel and t is time. This equation is something of a rarity in hydraulics, as it is exact. No analytical approximations have been introduced in the derivation, other than that the channel is straight. (For curved channels the corresponding equation has been obtained by Fenton and Nalder (1995)). Some previous derivations have obtained the $\partial Q/\partial x$ term by assuming that the streamwise velocity component is constant over the section, which is not necessary. The simplicity of equation (1) and the fact that it is exact strongly suggest that in the mass-conservation equation at least, discharge and cross-sectional area are more fundamental than velocity and surface elevation, which have been the quantities often used in the past.

Now if we consider the integral form of the *momentum equation*, it can be shown that (for example, Fenton, 1999**?**):

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\int_A u^2\, dA + \frac{1}{\rho}\int_A \frac{\partial p}{\partial x}\, dA + gAS_f = qu_q. \tag{2}$$

where u is the streamwise velocity at any point, p is the pressure, ρ is the fluid density, g is the acceleration due to gravity, S_f is the friction slope with the usual convention that a positive friction slope corresponds to a force directed upstream, and u_q is the velocity of the inflow before mixing. The integrals are to be evaluated over the whole cross-section of the flow. This equation also contains no essential analytical approximations, although the frictional term is usually approximated by a crude but convenient empirical expression, either Chézy or Manning's equation. The two equations (1) and (2) are the fundamental equations governing the propagation of disturbances. They are both exact, although the latter is in integral form, and to evaluate it for practical use it is necessary to insert expressions for velocity u and pressure p, which we will now do.

Boussinesq equations

The appellation "Boussinesq equations" is a generic one for long wave equations which contain the effects of wave height and length to first order, which are widely used in coastal engineering, (for example, Peregrine, 1972). It is surprising that they have not been used more in hydraulics. Teng and Wu (1992) have obtained channel Boussinesq equations for the case of irrotational flow and no friction. Here we obtain Boussinesq equations for real rivers and channels with friction and which contain the effects of non-uniformity of velocity distribution and non-hydrostatic pressure distribution to first order.

It has been shown (Fenton, 1999**?**) that the integrals in equation (2) may be approximated with more accuracy than has generally been recognised. The approximation

$$\int_A u^2\, dA \approx \frac{Q^2}{A} \tag{3}$$

is actually exact for a velocity distribution which is linear both across the channel and vertically, which would conveniently albeit roughly describe many flows in nature. If we are satisfied with that approximation there is no need to introduce any momentum coefficients. In the absence of any detailed knowledge of the velocity distribution, this is about as far as can be gone.

Now we consider the pressure term

$$\frac{1}{\rho}\int_A \frac{\partial p}{\partial x} dA. \tag{4}$$

The conventional approach in hydraulics is to use the hydrostatic approximation, that the pressure at a point is given by the equivalent static head of fluid above. Here we consider a higher level of approximation. If we consider the equations of motion of an inviscid fluid where the wave motion is long then it can be shown (Peregrine, 1972) that the pressure gradients in the fluid in directions y and z, transverse to the channel and vertically upwards respectively, are given approximately by:

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -\frac{\partial v}{\partial t} \quad \text{and} \quad \frac{1}{\rho}\frac{\partial p}{\partial z} = -g - \frac{\partial w}{\partial t}, \tag{5}$$

where (v, w) are the velocity components in directions (y, z). The significance of these equations is that the pressure gradients are given by the local fluid acceleration, modified by g in the case of the vertical component. The hydrostatic approximation includes only the latter term and neglects the fluid acceleration terms. If it is assumed that the vertical acceleration term varies linearly between the surface and the bed, it can be shown after some manipulations, including equation (3) substituted into equation

(2), that the momentum equation with A and Q as dependent variables is:

$$\frac{\partial Q}{\partial t}+2\frac{Q}{A}\frac{\partial Q}{\partial x}+\left(\frac{gA}{B}-\frac{Q^2}{A^2}\right)\frac{\partial A}{\partial x}-\frac{2}{3}\frac{A\bar{d}}{B}\frac{\partial^3 Q}{\partial x^2\partial t}+gA\left(S_f-\bar{S}\right)=qu_q, \tag{6}$$

where B is the width of the surface at any section, $\bar{d}$ is the depth of the centroid of the cross-section below the surface, which could well be assumed to be a representative constant value, and $\bar{S}$ is the mean bed slope of the stream. Hence we have equations (1) and (6), a pair of partial differential equations with area A and discharge Q as dependent variables and distance x and time t as independent variables. They describe the propagation of long waves and flows in streams where the effects of non-uniformity of velocity and pressure distributions are included to first order. In fact, those effects have manifested themselves as a single third derivative term in equation (6), which we will refer to as the dispersive term for reasons which will become obvious.

Behaviour of solutions of the Boussinesq equations

We consider a situation where disturbances are small perturbations about a uniform flow of velocity U_0 and area A_0. Here we wish to examine the effect of the dispersive term, and so for simplicity we will ignore effects of inflow, friction and bed slope. It can be shown that equations (1) and (6) become

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0, \tag{7}$$

$$\frac{\partial Q}{\partial t}+2U_0\frac{\partial Q}{\partial x}+\left(c_0^2-U_0^2\right)\frac{\partial A}{\partial x}-\frac{D_0^2}{3}\frac{\partial^3 Q}{\partial x^2\partial t}=0, \tag{8}$$

where $c_0=\sqrt{gA_0/B_0}$, commonly referred to as the speed of propagation of long waves (which we will see is not necessarily the case) and where $D_0^2=2A_0\bar{d}_0/B_0$, such that in a rectangular channel D_0 is the actual depth.

We assume that the wave can be Fourier decomposed into a number of sinusoidal waves, any one of which may be considered to vary like $\exp\left(ik\left(x-ct\right)\right)$, where $i=\sqrt{-1}$, corresponding to a sinusoidal wave of length $2\pi/k$ travelling at speed c. It is simpler here to restrict attention to the case where there is no underlying flow, such that $U_0=0$. Letting $A=A_0+A_1e^{ik(x-ct)}$, and $Q=Q_1e^{ik(x-ct)}$, we can show that there is a solution to the system of equations (7) and (8) only if the wave speed is given by

$$c^2=\frac{c_0^2}{1+\frac{1}{3}k^2D_0^2}. \tag{9}$$

This shows the effects of dispersion: that, instead of the traditional result that the wave speed is independent of length, such that $c^2=c_0^2=gA_0/B_0$, the wave speed now depends on the wavelength of disturbances, as given by the wavenumber k.

The result (9) fits in with traditional linear water wave theory for two-dimensional disturbances moving over stationary fluid whose motion remains irrotational, where the vertical distribution of velocity may be incorporated exactly. In that case, where the water is of depth d, the well-known result is (see Peregrine, 1972, for example):

$$\frac{c^2}{c_0^2}=\frac{\tanh kd}{kd}, \tag{10}$$

and taking the power series approximation of this to second order we obtain

$$\frac{c^2}{c_0^2}=1-\frac{1}{3}k^2d^2+\ldots, \tag{11}$$

which agrees with equation (9) when the power series expansion of that expression is calculated for the case of a rectangular channel or a two-dimensional flow, when $D_0=d$.

The above work agrees with that of Teng and Wu (1992), who solved the case of waves in a channel where there is no friction and where the flow is irrotational. In their case, using surface elevation and supposed mean fluid velocity as dependent variables, it was necessary to solve an elliptic partial differential equation to obtain the dimensionless quantity κ in the coefficient of their dispersion term, which they expressed as $-\frac{1}{3}\kappa^2 \left(A_0/B_0\right)^2 \bar{u}_{xxt}$, where they obtained solutions for some elementary channel cross-sections. In the present work the dispersion term is obtained explicitly in terms of the channel cross-section: $-\frac{2}{3}A_0\bar{d}_0/B_0 Q_{xxt}$. To compare Teng and Wu's result with the present work, it can be seen that κ^2 plays the same role as $2B_0\bar{d}_0/A_0$, or, 2 × (depth of centroid / mean depth). In Table 1 we show a comparison between the values for four elementary sections. It can be seen that disagreement is not large, which is perhaps surprising considering the two very different approaches.

Channel section	κ (Teng & Wu)	$\sqrt{2B_0\bar{d}_0/A_0}$
rectangular	1	1
semicircular	1.06	1.04
parabolic	1.16	1.10
triangular	1.27	1.15

Table 1. Numerical coefficients in dispersion terms from Teng & Wu and the present work.

In considering the full equations, however, the system is nonlinear (as the coefficients are actually functions of the dependent variables) in addition to being dispersive. This can be quantified roughly by interpreting equation (9) in a nonlinear sense, replacing the reference uniform flow values by the actual ones, and using a power series expansion consistent with the long wave approximation, giving

$$c^2 = \frac{gA/B}{1+\frac{2}{3}k^2 A\bar{d}/B} = \frac{gA}{B}\left(1-\frac{2}{3}k^2\frac{A\bar{d}}{B}+\ldots\right). \tag{12}$$

whose main effect is that higher disturbances travel faster, as the leading term A/B is the mean depth, and increasing depth means a larger value of c which is familiar from hydraulic theory. The dispersive term works such that longer waves (small k) are the fastest. In the propagation of arbitrary waves, the two effects will interact.

The Saint-Venant equations

Now we consider the behaviour of the traditional hydraulic approximation, where there is no dispersive term. To do this we simply delete the dispersive term from equation (6) to give the Saint-Venant equations in the form where the dependent variables are the cross-sectional area A and discharge Q, as derived and presented by Fenton (1999**?**):

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}-q=0, \text{ and} \tag{13}$$

$$\frac{\partial Q}{\partial t}+\left(\frac{gA}{B}-\frac{Q^2}{A^2}\right)\frac{\partial A}{\partial x}+\frac{2Q}{A}\frac{\partial Q}{\partial x}+gA\left(S_f-\bar{S}\right)-qu_q=0. \tag{14}$$

The traditional analysis of these equations, aided by use of the method of characteristics (see Stoker, 1957, for example) shows that the gradient of the characteristics is

$$\frac{dx}{dt}=\frac{Q}{A}\pm\sqrt{\frac{gA}{B}}, \tag{15}$$

such that there is a current velocity Q/A, on which waves travel upstream and downstream at a rate $\pm\sqrt{gA/B}$. The traditional interpretation is that waves also travel at these velocities at which information travels up and down the channel. The role of the frictional terms has been assumed to be simply to modify the magnitude of the waves, which are supposed to show no dispersive behaviour. In the next section we will show that a proper analysis of the equations, incorporating the friction terms, shows that the solutions of the St Venant equations are in fact, not simply dissipative, but are in fact diffusive and dispersive as well. To do this we will consider a linearisation of the equations, but not done as simply as that for the Boussinesq equations without friction as shown above.

Behaviour of solutions of the Saint-Venant equations

We consider small perturbations about a uniform flow, with no inflow, $q = 0$. Let the friction slope be written in the general form $S_f = \beta U \left|U\right|^{n-1}/R^m$, where β is a friction coefficient, $U = Q/A$ is the mean velocity in the channel, and $R = A/P$ is the hydraulic radius, where P is the wetted perimeter. In the case of Manning's law, $m = 4/3$, for Chézy's law $m = 1$, and for both laws $n = 2$. We linearise the friction term by expanding about a reference flow of area A_0, discharge Q_0, and wetted perimeter P_0. It is convenient to introduce S, the common slope corresponding to the bed slope and surface slope of the reference flow. Substituting into the St Venant equations, taking only first order terms, and substituting back for the perturbed quantities, the St Venant equations with the linearised forcing terms may be written:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0, \quad \text{and} \tag{16}$$

$$\frac{\partial Q}{\partial t} + \left(c_0^2 - U_0^2\right)\frac{\partial A}{\partial x} + 2U_0\frac{\partial Q}{\partial x} + \Theta_1\left(A - A_0\right) + \Theta_2\left(Q - Q_0\right) = 0, \tag{17}$$

where c_0 and U_0 are as previously. The coefficients Θ_1 and Θ_2 express the effects of friction as

$$\Theta_1 = -gS\left(n + m\left(1 - \frac{A_0 P_0'(A_0)}{P_0}\right)\right), \tag{18}$$

$$\Theta_2 = ngS/U. \tag{19}$$

We now examine the nature of solutions of these equations, by seeking solutions which vary like $\exp\left(ikx - \mu t\right)$, where k is the wavenumber such that the variation is periodic in x with a wavelength $2\pi/k$ and where variation in time is like $e^{-\mu t}$. It is the nature of the coefficient μ which determines the behaviour of the waves, whether they grow or decay, and how fast they travel. Let $A = A_0 + A_1 e^{ikx-\mu t}$ and $Q = Q_0 + Q_1 e^{ikx-\mu t}$, where A_1 and Q_1 are constants, the Fourier coefficients of the wave of wavenumber k. Substituting into equations it can be shown that for solutions to exist

$$\mu = \underbrace{\alpha - \frac{s}{\sqrt{2}}\sqrt{\phi + \alpha^2 - k^2c_0^2}}_{\text{DECAY RATE}} + \underbrace{ik\left(U_0 + \frac{s}{k\sqrt{2}}\sqrt{\phi - \alpha^2 + k^2c_0^2}\right)}_{\text{PROPAGATION VELOCITY}}, \tag{20}$$

where s is either $+1$ or -1 throughout any one equation, and where we use ϕ to denote the term

$$\phi = \sqrt{\left(k^2c_0^2 - \alpha^2\right)^2 + 4\alpha^2k^2V^2}. \tag{21}$$

As shown, the real part is the rate at which the amplitude of the wave decays, and the imaginary part in brackets is proportional to the velocity of propagation. As solutions with both $s = \pm 1$ are possible, we see that we have waves propagating both up and down the stream, each with different values of the decay constant.

What is important is that both the decay rate and the propagation velocity, as shown in equation (20), depend on the wavenumber k, such that the former is actually a diffusive process and the latter is one of dispersion, as distinct from traditional interpretations of the St Venant equations. Taking that part of the propagation velocity in equation (20) which is added on to the velocity of the stream U_0, we can interpret this as the speed of waves relative to the water, which can be written

$$\frac{c}{c_0} = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{\alpha^2}{k^2c_0^2} + \sqrt{\left(1 - \frac{\alpha^2}{k^2c_0^2}\right)^2 + 4\frac{V^2}{c_0^2}\frac{\alpha^2}{k^2c_0^2}}}, \tag{22}$$

having substituted for ϕ from equation (21). It is difficult to make deductions from this complicated expression, but we can see that the wave speed depends on only two parameters, α/kc_0 and V/c_0. The latter is the Vedernikov Number:

$$\mathbf{V} = \frac{V}{c_0} = \frac{U}{c_0}\frac{m}{n}\left(1 - \frac{A_0}{P_0}\frac{dP_0}{dA_0}\right). \tag{23}$$

It is more revealing to examine the behaviour of equation (22) in each of two limits, for both large and small values of α/kc_0. We expand equation (22) as a power series in this quantity and we get

$$\frac{c}{c_0} = 1 - \frac{1}{2}(1-\mathbf{V}^2)\left(\frac{\alpha}{kc_0}\right)^2 + O\left(\left(\frac{\alpha}{kc_0}\right)^4\right), \tag{24}$$

showing that in the limit of no friction as $\alpha/kc_0 \to 0$, then $c \to c_0$, and we do have the traditional result that waves propagate at $c_0 = \sqrt{gA_0/B_0}$.

In the case where the flow is dominated by friction, we expand equation (22) as a power series in kc_0/α:

$$c = V\left(1 + \frac{1}{2}\left(1-\mathbf{V}^2\right)\left(\frac{kc_0}{\alpha}\right)^2 + O\left(\left(\frac{kc_0}{\alpha}\right)^4\right) + \ldots\right). \tag{25}$$

This result shows that to first order the waves travel at a speed of $c = V$, which is governed by the velocity of the water in the channel as given by equation (23), rather than travelling at the speed of long waves on still water, c_0.

In general, the waves will be free to travel at the speed given by equation (22), which will be somewhere between the two values given by equations (24) and (25) in the two limits of small and large friction. Generally, if there is any friction, the speed depends on the wavelength of disturbances and the system is dispersive. The results show that the nature of wave propagation in a waterway is more complicated than conventional long wave theory suggests, and the speed of propagation of simple disturbances may not be given by the traditional formula.

Conclusions

The principal equations for describing the movement of long waves and floods in waterways have been derived. Throughout, the integrated quantities of cross-sectional area and discharge have been used as variables rather than surface elevation and velocity, as they are more fundamental, and the equations can be derived with surprisingly few assumptions. A new higher-order Boussinesq approximation has been obtained which neither assumes constant streamwise velocity nor makes the traditional hydrostatic approximation. This formulation may prove useful in practice for the computation of waves and surges. Approximate solutions have been obtained which show that wave speed depends on wavelength, which can lead to effects not predicted by the conventional long wave equations, such as the development of an undular bore.

By deleting a single term corresponding to the non-hydrostatic contribution, a new form of the St Venant long wave equations has been obtained. Traditional interpretations of these equations are that all disturbances travel at a speed given only by the depth of flow. Here, it has been shown that the behaviour of solutions of these equations is actually rather more complicated. Solutions show wavelike behaviour, but they also show diffusive and dispersive behaviour, where the decay and propagation behaviour of the solutions show dependence on wavelength. A novel inference from this is that the velocity of propagation of waves in channels with friction is not the widely-presumed long wave speed given by the local mean depth. These results suggest that the St Venant equations require more than the traditional interpretations.

References

Cunge, J. A., Holly, F. M. & Verwey, A. (1980) *Practical Aspects of Computational River Hydraulics*, Pitman, London.

Fenton, J. D. & Nalder, G. V. (1995) Long wave equations for waterways curved in plan, *in Proc. 26th Congress IAHR, London*, Vol. 1, pp. 573–578.

Peregrine, D. H. (1972) Equations for water waves and the approximation behind them, *Waves on Beaches and Resulting Sediment Transport*, R. E. Meyer (ed.), Academic, New York.

Stoker, J. J. (1957) *Water Waves*, Academic.

Teng, M. H. & Wu, T. Y. (1992) Nonlinear water waves in channels of arbitrary shape, *J. Fluid Mech.* **242**, 211–233.