and the spatial formulations of the linear stability problem will lead to opposite conclusions. The present analysis for the case of sinuous disturbances offer an excellent example. The same equation also implies that both temporal and spatial formulations will lead to the same conclusion on the stability criteria if the group velocity is in the direction of the flow. Our analysis for varicose disturbances is a good example. Another example is furnished by the stability problem of a liquid layer flow down an inclined plane. The equivalence of the temporal and spatial formulations for this problem has been discussed recently by Lin (1975) and Krantz (1975).

It should be pointed out that we neglected terms of $O(\delta)$ in the analysis. Thus our results based on the quasi-parallel flow approximation are valid first order solutions which predict the first order effects in a liquid curtain with sufficient accuracy only if its thickness is so thin that $\delta \ll 1$. For such a liquid curtain the omitted higher order terms in the linear analysis may well be less significant than the neglected nonlinear effects. It will be of interest to know if the conclusions on stability reached by the present normal mode analysis will be altered by the findings from the corresponding stability analysis in the frame work of initial value problems. Although more general disturbances which can be constructed from normal modes by Fourier superposition will give the same stability criteria, the disturbance corresponding to any continuous spectrum in the initial value problem may not. However, based on the good agreement between our theory and known experiments, we conjecture that the transient part of the solution to the initial value problem will be damped and the normal mode solution recovered.

It is seen from (18) that the speed of sinuous disturbances decreases as the curtain thickness increases. Therefore the disturbances which propagate upstream will experience overturning when they are forced to overtake the waves in front of them, since the curtain thickness increases in the upstream direction. It is very unlikely that one will find supercritical stability in the nonlinear analysis. However, there is an evidence of sub-critical instability. Brown observed that if the disturbance amplitude is so large as to cause the two free surfaces to meet the curtain will break, even if $W < W_c$, to form an inverted V-shaped free edge. G. I. Taylor (1959*b*) in fact demonstrated that $W_c = \frac{1}{2}$ from a momentum balance for an element of such a free edge in a broken sheet of an inviscid liquid.

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A Fourier approximation method for steady water waves

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A method for the numerical solution of steadily progressing periodic waves on irrotational flow over a horizontal bed is presented. No analytical approximations are made. A finite Fourier series, similar to Dean's stream function series, is used to give a set of nonlinear equations which can be solved using Newton's method. Application to laboratory and field situations is emphasized throughout. When compared with known results for wave speed, results from the method agree closely. Results for fluid velocities are compared with experiment and agreement found to be good, unlike results from analytical theories for high waves.

The problem of shoaling waves can conveniently be studied using the present method because of its validity for all wavelengths except the solitary wave limit, using the conventional first-order approximation that on a sloping bottom the waves at any depth act as if the bed were horizontal. Wave period, energy flux and mass flux are conserved. Comparisons with experimental results show good agreement.

1. Introduction

Traditionally there have been two main approaches to the nonlinear problem of a train of waves of constant form propagating steadily over fluid on a horizontal bed, both based on expansions in a small parameter. The best known is that of Stokes in which this parameter is the leading coefficient of a Fourier expansion, appearing in dimensionless form as $a_1 k$ where k is the wavenumber. This expansion has been taken to fifth order by De (1955) and Skjelbreia & Hendrickson (1961), and similar expansions to very high order by Schwartz (1974) and Cokelet (1977). These high-order expansions made extensive use of computer manipulation of the series and necessary convergence improvement techniques to obtain accurate results. Examination of the fifth-order results shows that the expansion parameter is effectively $a_1 k/\sinh^3 k\overline{\eta}$, where $\overline{\eta}$ is the mean water depth. Hence, for these Stokes expansions to be rapidly convergent, $a_1 k$ should be small and the water depth $\overline{\eta}$ be large enough so that the necessary condition $a_1 k \ll (k\overline{\eta})^3$ is also satisfied. Thus Stokes' approach is best suited to waves which are not too high in water which is not too shallow.

Complementary to Stokes' expansions are those which are based on series in terms of shallowness ($\overline{\eta}/\lambda$, where λ is the wavelength), giving rise to cnoidal wave solutions. When these series are recast in terms of wave height H, it is found that the effective expansion parameter is $H/m\overline{\eta}$, where m is the modulus of the elliptic functions which occur throughout cnoidal wave theory and which becomes small for waves in deep water (see Fenton 1979). For cnoidal wave expansions to give accurate results, $H/\overline{\eta}$ must be small. The requirement that m is not small must also hold and this is easily shown to correspond to the condition $(\bar{\eta}/\lambda)^2 \ll 1$. Cnoidal wave solutions are thus applicable to waves which are not too high in water which is not too deep. In any practical application it is desirable to use whichever of the two theories is appropriate to the water depth: neither approach is uniformly valid in all water depths.

Of the studies mentioned above, few have concentrated on presenting results in a directly applicable form. Exceptions include the fifth-order Stokes wave solution of Skjelbreia & Hendrickson (1961) and the fifth-order cnoidal wave solution of Fenton (1979). In their respective regions of validity, the series expansions give good results for overall wave train parameters such as the wave speed; however, for details of the flow field, such as fluid velocity, they are not always satisfactory: for example, for comparison with the experimental results for long waves of Le Méhauté, Divoky & Lin (1968), fifth-order Stokes wave solutions could only be obtained for one quarter of the experimental cases – the shortest waves. On the other hand, cnoidal wave theory in the long-wave region gave excellent results for waves of height $H/\bar{\eta} \approx 0{\cdot}4$, but for $H/\bar{\eta} \approx 0{\cdot}5$ was considerably in error. In view of this lack of uniform accuracy and validity, there is a need for a reliable method for solving steady wave problems in a form which makes no essential analytical approximations, which is valid for both deep and shallow water and which is capable of direct application.

A method which has the potential for satisfying these criteria is that developed by Chappelear (1961) and the 'stream function' method of Dean (1965, 1970*a*, *b*), which may be categorized as 'Fourier approximation methods', as they are based on the use of truncated Fourier expansions for field quantities. Assuming such an expansion so as to satisfy the field equation and bottom boundary condition identically, the problem reduces to solving a number of nonlinear equations for each of the Fourier coefficients, for equi-spaced values of the surface elevation and for quantities characterizing the wave train as a whole, such as wave speed. In the approach of Chappelear and Dean, solution of these equations proceeded by a method of successive corrections to an initial estimate in such a way that the least-squares errors in the surface boundary conditions were minimized.

There are several aspects of this approach and its reported results which are not altogether satisfactory, and which may have inhibited its deserved widespread use. With the technique as set up by previous workers, truncation of the series is not the only approximation; a limitation to the attainable accuracy has been introduced by the use of a Simpson's rule integration at one stage in the solution process. An alternative version of Dean's method (using a Schmidt orthogonalization process) has been given by Chaplin (1980), who obtained results which agreed more closely with the trend of Cokelet's results. The method, like the original version, is not straightforward in application, and neither of them can be used for waves in deep water since the stream function expansions contain hyperbolic functions which become very large for this case. These methods do not allow for the possible specification of mass flux as determining the wave speed, a situation which is commonly encountered.

As pointed out by Stokes (1847) waves can travel at any speed without change of form. In any particular frame of reference the waves travel at a speed determined by some quantity such as current speed or mass flux *in that frame*, as well as constraints due to viscosity such as the relative motion between wave and bottom. If viscosity is neglected, the wave travels at a speed relative to the bottom determined by some specified value of mean current or mass flux. It is possible to solve the problem in a frame relative to which motion is steady, without having to define the wave speed. However, in most situations the waves are viewed from a different frame of reference in which they are not stationary. If the wave period in this frame is specified, or if the fluid velocities in this frame are required, an assumption as to wave speed must be included in the analysis. The case where the time mean mass transport throughout the fluid is specified applies to, amongst other situations, wave tank experiments such as those of Le Méhauté *et al.* (1968) where the end of the tank is closed and mass flux is zero. Dean (1974) has produced extensive tables of integral quantities and fluid velocities for a number of particular cases of wave height and period. However, all these seem to be for the special case of when the Eulerian mean velocity is zero, that is, the current is zero.

In view of these comments on the lack of universal applicability of Stokes and cnoidal wave expansions and on some unsatisfactory aspects of the stream function method, it was decided to develop a numerical method, also based on Fourier approximation techniques, having as its only approximation the truncation of the Fourier series. It would be valid for deep and shallow water (but not for the solitary wave limit) and would be flexible enough for any two quantities (such as wave height and period, or wave period and energy flux) to be specified so that a solution could be obtained. The development of such a method, using Newton's technique for the solution of a system of nonlinear equations, is described in § 2. In § 3, results from the method are compared with experimental results for the velocity profile under the crest of periodic long waves. Because of the ease of application of the method set up in § 2 to a practical situation where any current or mass transport speed, wave height and period may be specified and velocities, accelerations etc. determined at any point in the fluid, it was considered unnecessary to produce tables for limited situations from which the desired quantities would have to be found by interpolation. Finally, in § 4, the problem of shoaling waves is studied. The present method is not strictly applicable to this problem, but its accuracy for waves over a horizontal bed in water of almost any depth make it more accurate and widely applicable than previous shoaling approximations.

2. The steady wave equations and their solution

2.1. *A Fourier approximation to the equations*

The problem considered is that of two-dimensional periodic waves propagating without change of form over a layer of fluid on a horizontal bed. With horizontal co-ordinate x and vertical co-ordinate y, the origin is on the bed and moves with the same velocity as the waves so that in this frame of reference all motion is steady. If the fluid is incompressible it is possible to define a stream function $\psi(x,y)$ such that the velocity components (u,v) are given by

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x,$$

and if the motion is irrotational, ψ satisfies Laplace's equation throughout the fluid:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0. \tag{1}$$

The boundary conditions to be satisfied are

$$\psi(x,0) = 0, \quad \psi(x,\eta(x)) = -Q, \qquad (2), (3)$$

where $y = \eta(x)$ on the free surface and Q is a positive constant denoting the total volume rate of flow underneath the steady wave per unit length in a direction normal to the x, y plane (henceforth referred to as unit span). With this sign convention the apparent flow is from right to left, in a negative-x direction. On the free surface, the pressure is constant so that Bernoulli's equation gives

$$\frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right]+\eta = R, \tag{4}$$

where R is a constant. In these equations, all variables have been non-dimensionalized with respect to the average depth, $\bar{\eta}$, and gravitational acceleration, g: that is, x is used for $x/\bar{\eta}$, y for $y/\bar{\eta}$, η for $\eta/\bar{\eta}$, ψ for $\psi/(g\bar{\eta}^3)^{\frac{1}{2}}$, Q for $Q/(g\bar{\eta}^3)^{\frac{1}{2}}$ and R for $R/g\bar{\eta}$. Other non-dimensionalized variables to be introduced are: the wave speed c for $c/(g\bar{\eta})^{\frac{1}{2}}$; the wavenumber k for $k\bar{\eta} = 2\pi\bar{\eta}/\lambda$ where λ is the wavelength; the wave period τ for $\tau(g/\bar{\eta})^{\frac{1}{2}}$; and an arbitrary reference level D for $D/\bar{\eta}$.

If the symmetry of the wave about the crest is exploited, a series for $\psi(x, y)$ can be written

$$\psi(x,y) = B_0 y + \sum_{j=1}^{N} B_j \frac{\sinh jky}{\cosh jkD}\cos jkx, \tag{5}$$

satisfying equation (1) and the boundary condition (2). The $B_0, \ldots, B_N$ are constant for a particular wave. The assumption, for computational purposes, that N is finite is the only approximation made in this method of solution.

Thus equation (3) becomes

$$B_0\eta + \sum_{j=1}^{N} B_j \frac{\sinh jk\eta}{\cosh jkD}\cos jkx = -Q \quad \text{for all} \quad x, \tag{6}$$

and (4) becomes

$$\frac{1}{2}\left\{B_0 + k\sum_{j=1}^{N} jB_j \frac{\cosh jk\eta}{\cosh jkD}\cos jkx\right\}^2 + \frac{1}{2}\left\{k\sum_{j=1}^{N} jB_j\frac{\sinh jk\eta}{\cosh jkD}\sin jkx\right\}^2 + \eta = R \quad \text{for all} \quad x. \tag{7}$$

In his approach, Chappelear (1961) introduced a Fourier series for $\eta(x)$; it is clearly simpler to use $\eta(x)$ itself in these equations.

It was found that the exponential behaviour of $\sinh jk\eta$ and $\cosh jk\eta$, for large values of j, caused undesirable numerical errors without the $\cosh jkD$ term in the denominator and so this compensating factor was introduced, thus redefining the Fourier coefficients. For large values of $|j|$,

$$\frac{\cosh jk\eta}{\cosh jkD} \sim \frac{\sinh jk\eta}{\cosh jkD} \sim \exp\left[|j|k(\eta - D)\right],$$

and so if D is chosen in the range of η or slightly greater than the maximum value of η, any problems associated with these hyperbolic functions are surmounted. Since its value is somewhat arbitrary, perhaps the best choice is $D = 1$, corresponding to the mean depth. With this introduction of $\cosh jkD$ in the denominator, waves in deep water, possibly infinitely deep, can be studied.

To solve the problem numerically, equations (6) and (7) are to be satisfied at $2N$ points equally spaced over one wavelength, though by symmetry only $N+1$ points, from the wave crest to the trough, need to be considered.

Let $\eta_m = \eta(x_m)$ where $x_m = m\lambda/2N$, $m = 0, 1, \ldots, N$ and $kx_m = \pi m/N$, so that equations (6) and (7) become

$$B_0\eta_m + \sum_{j=1}^{N} B_j \frac{\sinh jk\eta_m}{\cosh jkD}\cos(jm\pi/N) + Q = 0 \quad \text{for} \quad m = 0, 1, \ldots, N, \tag{8}$$

and

$$\tfrac{1}{2}u_m^2 + \tfrac{1}{2}v_m^2 + \eta_m - R = 0, \tag{9}$$

also for $m = 0, 1, \ldots, N$, where

$$u_m = B_0 + k\sum_{j=1}^{N} jB_j \frac{\cosh jk\eta_m}{\cosh jkD}\cos(jm\pi/N) = u(x_m, \eta_m),$$

and

$$v_m = k\sum_{j=1}^{N} jB_j \frac{\sinh jk\eta_m}{\cosh jkD}\sin(jm\pi/N) = v(x_m, \eta_m).$$

These $2N+2$ nonlinear equations involve the $2N+5$ variables η_j, B_j $(j = 0, \ldots, N)$, k, Q and R. To obtain a solution 3 more equations must be specified. Since variables have been non-dimensionalized with respect to $\bar{\eta}$, an equation for the unit mean depth may be written

$$\frac{1}{2N}\left[\eta_0 + \eta_N + 2\sum_{j=1}^{N-1}\eta_j\right] - 1 = 0. \tag{10}$$

This simple trapezoidal rule for the numerical integration of the periodic function η can be shown to be of the same accuracy as the previous equations by writing η as an N-term Fourier series and performing some simple manipulations.

With the values of R and Q specified, these equations may be solved for the remaining unknowns. However, for practical problems, it is usually values of the wave height H, and the period τ, which define the problem. Two additional equations which specify these physical parameters are

$$\eta_0 - \eta_N - H = 0, \tag{11}$$

where η_0 is the surface elevation at the crest and η_N that at the trough and

$$kc\tau - 2\pi = 0. \tag{12}$$

The latter equation introduces one more variable, namely, the wave speed c. As mentioned in § 1, the assumption as to the speed at which a wave travels must be stated. This may be done by specifying the time mean Eulerian velocity c_E throughout the fluid. In the steady frame the mean velocity at each level within the fluid is B_0, which is negative. To consider motion relative to any frame through which the waves move, a uniform wave speed c is superimposed, so that in this frame, the time mean Eulerian velocity $c_E = c + B_0$. Thus, if the current c_E is specified, c satisfies the equation

$$c - c_E + B_0 = 0. \tag{13a}$$

Alternatively, it may be appropriate to specify the mean particle drift velocity (that is, the mass transport velocity) c_s. In the steady frame with mean fluid depth unity, the volume rate of flow, equal to the mean velocity with which particles move under the wave, is $-Q$. Therefore, in another frame the mass transport velocity $c_s = c - Q$, so that if c_s is specified, c satisfies

$$c - c_s - Q = 0. \tag{13b}$$

This method does not rely on the specification of H and τ – any other two variables may be assigned in their place. However, if τ is specified, then by virtue of (12) some value of either c_E or c_s must be given to be used in (13*a*) or (*b*).

The $2N+6$ equations (8)–(13) form a closed system for the unknown variables η_j, B_j $(j = 0, \ldots, N)$, c, k, Q and R.

2.2. *Solution by Newton's method*

The system of nonlinear equations (8)–(13) may be written

$$f_i(\eta_j, B_j(j = 0, \ldots, N), c, k, Q, R) = 0, \quad i = 1, \ldots, 2N+6, \tag{14}$$

where for $i = 1, \ldots, N+1$, the f_i represent (8), for $i = N+2, \ldots, 2N+2$, the f_i represent (9), f_{2N+3} is (10), f_{2N+4} is (11), f_{2N+5} is (12) and f_{2N+6} is either (13*a*) or (*b*). This set of equations can be solved by a Newton's method which iterates, with quadratic convergence, to a solution from an initial approximation. If the system of equations (14) is written

$$f_i(\mathbf{z}) = 0, \quad i = 1, \ldots, 2N+6, \tag{15}$$

where $\mathbf{z} = \{z_l, l = 1, \ldots, 2N+6\}$ is the vector of arguments of f_i as in (14), and if the approximate solution vector after the nth iteration is $\mathbf{z}^n$, the error vector may be written

$$\mathbf{f}^n = \{f_i(\mathbf{z}^n), \quad i = 1, \ldots, 2N+6\} = \{f_i^n, i = 1, \ldots, 2N+6\}.$$

From a Taylor series expansion, the error at the next iteration will be

$$f_i^{n+1} = f_i^n + \sum_{l=1}^{2N+6} \left(\frac{\partial f_i}{\partial z_l}\right)^n (z_l^{n+1} - z_l^n) + \ldots, \quad i = 1, \ldots, 2N+6,$$

where $(\partial f_i/\partial z_l)^n = \partial f_i(\mathbf{z}_i^n)/\partial z_l$.

However, the desired result is $\mathbf{f}^{n+1} = 0$ and the solution vector $\mathbf{z}^{n+1}$ which approximately yields this result is found by truncating the series after the term shown and solving the resulting system of linear simultaneous equations written as a matrix equation:

$$\mathbf{A}(\mathbf{z}^{n+1} - \mathbf{z}^n) = -\mathbf{f}^n,$$

where

$$\mathbf{A} = [A_{il}], \quad A_{il} = \frac{\partial f_i}{\partial z_l}(\mathbf{z}^n).$$

The derivatives $\partial f_i/\partial z_l$ are obtained from equations (8)–(13) and are as follows, all derivatives not shown explicitly being zero.

For $i = 1, \ldots, N+1$, $m = i-1$;

$$\frac{\partial f_i}{\partial \eta_m} = u_m, \quad \text{where } u_m \text{ and } v_m \text{ are defined in (9);}$$

$$\frac{\partial f_i}{\partial B_0} = -\eta_m; \quad \frac{\partial f_i}{\partial B_j} = S_{jm}^{(1)}, \quad j = 1, \ldots, N;$$

$$\frac{\partial f_i}{\partial k} = \eta_m \frac{(u_m - B_0)}{k} - D \sum_{j=1}^{N} jB_j S_{jm}^{(1)} \tanh jkD$$

(it should be noted that $x_m = \pi m/Nk$, so $kx_m = \pi m/N$ is not a function of k); and

$$\frac{\partial f_i}{\partial Q} = 1.$$

For $i = N+2, \ldots, 2N+2$, $m = i-(N+2)$:

$$\frac{\partial f_i}{\partial \eta_m} = 1 + u_m k^2 \sum_{j=1}^{N} j^2 B_j S_{jm}^{(1)} + v_m k^2 \sum_{j=1}^{N} j^2 B_j C_{jm}^{(1)};$$

$$\frac{\partial f_i}{\partial B_0} = -u_m; \quad \frac{\partial f_i}{\partial B_j} = jku_m C_{jm}^{(2)} + jkv_m S_{jm}^{(2)}, \quad j = 1, \ldots, N;$$

$$\frac{\partial f_i}{\partial k} = u_m \left[(u_m - B_0)/k + k\eta_m \sum_{j=1}^{N} j^2 B_j S_{jm}^{(1)} - kD \sum_{j=1}^{N} j^2 B_j C_{jm}^{(2)} \tanh jkD\right]$$
$$+ v_m \left[v_m/k + k\eta_m \sum_{j=1}^{N} j^2 B_j C_{jm}^{(1)} - kD \sum_{j=1}^{N} j^2 B_j S_{jm}^{(2)} \tanh jkD\right];$$

$$\frac{\partial f_i}{\partial R} = -1.$$

Here,

$$\frac{\partial f_{2N+3}}{\partial \eta_j} = \begin{cases} 1/2N, & j = 0 \text{ and } j = N, \\ 1/N, & j = 1, \ldots, N-1, \end{cases}$$

$$\frac{\partial f_{2N+4}}{\partial \eta_j} = \begin{cases} 1, & j = 0, \\ -1, & j = N, \end{cases}$$

$$\frac{\partial f_{2N+5}}{\partial c} = k\tau, \quad \frac{\partial f_{2N+5}}{\partial k} = c\tau, \quad \frac{\partial f_{2N+6}}{\partial c} = 1.$$

For equation (13*a*), $\partial f_{2N+6}/\partial B_0 = 1$, while for (13*b*) $\partial f_{2N+6}/\partial Q = -1$. In the notation used above

$$S_{jm}^{(1)} = \frac{\sinh jk\eta_m}{\cosh jkD} \cos(jm\pi/N), \quad S_{jm}^{(2)} = \frac{\sinh jk\eta_m}{\cosh jkD} \sin(jm\pi/N),$$

$$C_{jm}^{(1)} = \frac{\cosh jk\eta_m}{\cosh jkD} \sin(jm\pi/N), \quad C_{jm}^{(2)} = \frac{\cosh jk\eta_m}{\cosh jkD} \cos(jm\pi/N),$$

The initial approximation to the solution is assumed to be a linear sinusoidal wave, that is,

$$\eta_m = 1 + \tfrac{1}{2}H \cos(m\pi/N) \quad \text{for} \quad m = 0, \ldots, N,$$

$$B_0 = -c, \quad B_1 = -\tfrac{1}{4}H/ck, \quad B_j = 0, \quad j = 2, \ldots, N,$$

$$R = 1 + \tfrac{1}{2}c^2, \quad Q = c,$$

where c and k are found recursively from

$$k = 2\pi/\tau c, \quad c = \{\tanh k/k\}^{\frac{1}{2}},$$

with an initial guess $c = 1$, corresponding to a long-wave approximation.

For shorter waves, with larger values of k, it was found that with a choice of $D = 0$, as used by Dean (1965), the iteration did not converge. This was because of the exponential behaviour of $\sinh jk\eta$ and $\cosh jk\eta$ as previously discussed. This problem was overcome by choosing $D = 1$.

The method was programmed and run on a computer. Convergence of the iteration was extremely rapid. When the method converged, the number of iterations was found to be independent of wave height, 5 iterations usually being sufficient for convergence to 12 decimal places. For very high waves it was found that the Stokes approximation was not a sufficiently accurate initial estimate of the solution and convergence was not achieved. In this situation it was necessary to extrapolate to the initial approximation from converged solutions for lower waves.

2.3. *Results for practical application*

The solution obtained by the above process may be used to provide results for other variables which may be of interest in areas of practical application.

(*a*) *Quantities varying with position and time.* All physical variables considered so far have been in a co-ordinate system which moves with the wave with the origin under the crest so that all motion in that frame of reference is steady. Accordingly, the steady velocities are

$$u(x,y) = B_0 + k \sum_{j=1}^{N} u_j(x,y),$$

where

$$u_j(x,y) = jB_j \frac{\cosh jky}{\cosh jkD} \cos jkx, \qquad (16a)$$

and

$$v(x,y) = k \sum_{j=1}^{N} v_j(x,y),$$

where

$$v_j(x,y) = jB_j \frac{\sinh jky}{\cosh jkD} \sin jkx. \qquad (16b)$$

Then, from Bernoulli's equation, the pressure $p(x,y)$ is given by

$$p(x,y) = R - y - \tfrac{1}{2}[u^2(x,y) + v^2(x,y)],$$

where $p(x,y)$ represents the dimensionless pressure $p(x,y)/\rho g\bar{\eta}$, and ρ is density.

Now in another co-ordinate system (X, Y) on the bed, in which motion is unsteady and the waves are moving from left to right at speed c, the unsteady velocities $U(X,Y,t)$, $V(X,Y,t)$ become

$$U(X,Y,t) = c + B_0 + k \sum_{j=1}^{N} u_j(X - X_0 - ct,\ Y),$$

$$V(X,Y,t) = k \sum_{j=1}^{N} v_j(X - X_0 - ct,\ Y),$$

where the u_j, v_j are defined in (16*a*), (16*b*). The wave crest is at $X = X_0$ when $t = 0$.

The derivatives are given by

$$\frac{\partial U}{\partial X} = -\frac{\partial V}{\partial Y} = -k^2 \sum_{j=1}^{N} j^2 B_j \frac{\cosh jky}{\cosh jkD} \sin jk\,(X - X_0 - ct),$$

$$\frac{\partial U}{\partial Y} = \frac{\partial V}{\partial X} = k^2 \sum_{j=1}^{N} j^2 B_j \frac{\sinh jky}{\cosh jkD} \cos jk(X - X_0 - ct),$$

$$\frac{\partial U}{\partial t} = -c\frac{\partial U}{\partial X}, \quad \frac{\partial V}{\partial t} = -c\frac{\partial V}{\partial X},$$

$$\frac{DU}{Dt} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y}, \quad \frac{DV}{Dt} = \frac{\partial V}{\partial t} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}.$$

Pressure is given by

$$P(X,Y,t) = R - Y - \tfrac{1}{2}[u^2(X - X_0 - ct,\ Y) + v^2(X - X_0 - ct,\ Y)].$$

(*b*) *Physical quantities characteristic of the wave train.* These quantities are introduced here in non-dimensionalized form, often using a mean over one wavelength or period which is denoted by an overbar. Several of the results of Longuet-Higgins (1975) and Cokelet (1977) are used to provide equations for the newly-introduced quantities in terms of the variables introduced in § 2.2. In fact, any of these relations could be used as part of the system of equations: for example, a numerical value of I might be specified and (17*a*) below would provide an equation to replace (11), (12) or (13).

Wave impulse: The mean wave impulse per unit horizontal area is

$$I = \overline{\int_0^\eta U\,dy}$$
$$= c - Q. \qquad (17a)$$

Kinetic energy: The mean kinetic energy per unit horizontal area is

$$T = \overline{\int_0^\eta \tfrac{1}{2}(U^2 + V^2)\,dy}$$
$$= \tfrac{1}{2}cI - (c + B_0)\,Q. \qquad (17b)$$

Potential energy: Mean potential energy due to the waves per unit horizontal area is

$$V = \overline{\int_{\bar{\eta}}^{\eta} (y - \bar{\eta})\,dy}$$
$$= \tfrac{1}{2}(\overline{\eta^2} - 1), \qquad (17c)$$

where

$$\overline{\eta^2} = \frac{1}{2N}\left[\eta_0^2 + \eta_N^2 + 2\sum_{j=1}^{N-1} \eta_j^2\right], \qquad (17d)$$

Mean square of bed velocity:

$$\overline{u_b^2} = \frac{1}{\lambda}\int_0^\lambda U^2(X,0,t)\,dX$$
$$= 2(R - 1) - c^2. \qquad (17e)$$

Radiation stress: The excess flux of momentum per unit span due to the waves is the radiation stress.

$$S_{xx} = \overline{\int_0^\eta (p + U^2)\,dy} - \tfrac{1}{2}$$
$$= 4T - 3V + \overline{u_b^2}. \qquad (17f)$$

Mean wave power: The mean wave power (or energy flux) per unit span is

$$F = \overline{\int_0^\eta [p+\tfrac{1}{2}(U^2+V^2)+(y-\eta)]\,U\,dy}$$

$$= (3T-2V)\,c+\tfrac{1}{2}\overline{u_b^2}(I+c)+c(c+B_0)\,Q. \qquad (17g)$$

Mean Stokes drift velocity:

$$c_s = c-Q = I. \qquad (17h)$$

Momentum flux: The momentum flux per unit span in the steady flow is

$$S = \int_0^\eta (p+U^2)\,dy$$

$$= S_{xx}-2cI+c^2+\tfrac{1}{2}.$$

In dimensional terms, I represents $I/\rho(g\bar{\eta}^3)^{\frac{1}{2}}$, T represents $T/\rho g\bar{\eta}^2$, V represents $V/\rho g\bar{\eta}^2$, $\overline{u_b^2}$ represents $\overline{u_b^2}/g\bar{\eta}$, S_{xx} represents $S_{xx}/\rho g\bar{\eta}^2$, F represents $F/\rho(g^3\bar{\eta}^5)^{\frac{1}{2}}$, c_s represents $c_s/(g\bar{\eta})^{\frac{1}{2}}$ and S represents $S/\rho g\bar{\eta}^2$.

2.4. *Accuracy of the solution*

The present method does not suffer from the disadvantages of the Stokes and cnoidal expansions in that it does not depend upon the waves being small and it is valid for all depths: it is essentially a numerical technique for the approximation of continuous, periodic functions by Fourier series. The theory of such series and knowledge of the behaviour of water waves provides some insight into the limits of the method.

It is well known that for higher waves the crest becomes more peaked until it approximates a sharp-crested wedge with a discontinuity of gradient. Any Fourier series for a function with a discontinuous first derivative has coefficients which decrease like n^{-2} which is much slower than for a function which is everywhere smooth. Thus, although the Fourier method makes no approximation as to wave height, the sharpness of the crest for higher waves means that larger values of N must be used to give accurate results. Also, longer waves tend to look like a solitary wave, that is, the elevation above mean depth is nontrivial only for a small portion of the wavelength. The associated Fourier series contain coefficients which oscillate and decay very slowly thus necessitating the use of larger values of N.

A numerical method for the solution of steady waves has been applied by Vanden-Broeck & Schwartz (1979). Its results support the accuracy of those of Cokelet (1977), and are capable of greater accuracy for longer waves. However the method is still of an inverse formulation and not convenient for practical application. Few results are given, whereas Cokelet has presented a number of results for the quantities given in § 2.3. It does not seem necessary to make detailed comparisons for all of these and since the wave speed c has traditionally been used as the first basis for comparison between wave theories, it is this quantity which is used to examine the accuracy of the present work. For each of ten different values of kQ/c, which is a measure of the ratio of wavelength to depth of fluid, Cokelet presented a table of corresponding values of his dimensionless variables $\frac{1}{2}kH$, kc^2 and $k(\bar{\eta}-Q/c)$. His results are indicated by the solid lines in figure 1 showing a plot of c^2 against H for each constant value of kQ/c. Each curve is labelled with the approximately constant value of the dimensionless wavelength λ since this quantity has more obvious physical meaning than kQ/c.

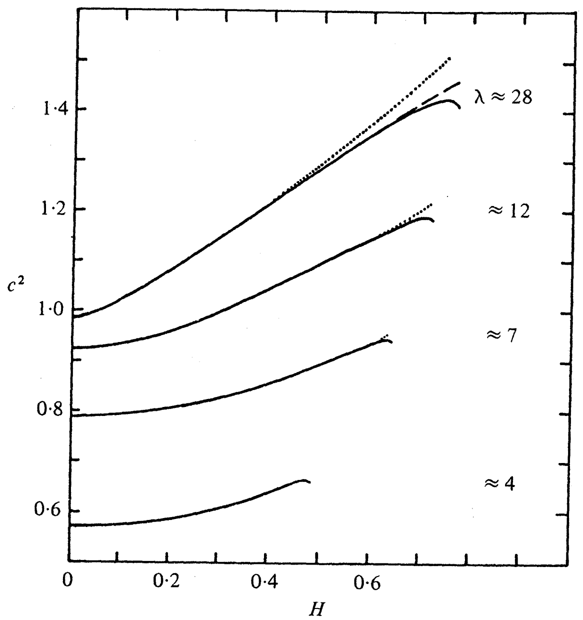

FIGURE 1. Comparison between the present method and the results of Cokelet (1977) for the wave speed squared, c^2, as a function of wave height H. Present method: $N = 8$ (· · ·), $N = 16$ (– – – –), $N = 32$ indistinguishable from Cokelet's results (——) at this scale. Each curve is drawn for a constant value of kQ/c taken from Cokelet's tables A2, A4, A6, A8, giving the almost constant values of wavelength shown.

To compare results from the present method with those of Cokelet, it was necessary to solve the problem, as set up in § 2.1, for the increasing values of H at a constant value of kQ/c as tabulated in his results. Thus, instead of specifying the period, the value $d\,(=kQ/c)$ was given and equation (12) replaced by

$$kQ-cd = 0.$$

Accordingly,

$$\partial f_{2N+5}/\partial Q = k, \quad \partial f_{2N+5}/\partial c = -d, \quad \partial f_{2N+5}/\partial k = Q$$

and the other derivatives are zero. Equation (13a) was used with $c_E = 0$, that is, at any depth, the mean Eulerian velocity over one period was zero.

As before, the initial approximations for the first two waves which have small amplitudes were found from linear theory. However, subsequently, in order to obtain a good initial solution, the estimate for each successive wave problem was determined by a linear extrapolation from the converged solutions for the previous two waves. Results are shown in figure 1 which indicates the very close agreement with Cokelet that was obtained so that the solutions are almost everywhere indistinguishable. Only for the very highest and longest waves with relatively coarse numerical approximation (small N) were significant errors obtained. For the longer waves larger values of N are required to give accurate results for the maximum in c^2 and beyond. Overall the numerical method based on Fourier approximation gave results which agreed closely with those from high-order Stokes series in which convergence improvement techniques were necessary.

$H/\bar{\eta}$	kc^2/g: Present method, $N = 16$	32	64	Cokelet	Vanden-Broeck & Schwartz
0·1729974	0·615059	0·615059	0·615059	0·615059	Not presented
0·2526308	0·631112	0·631112	0·631112	0·631112	Not presented
0·3802643	0·666501	0·666501	0·666501	0·666501	0·666501
0·4944549	0·706443	0·706443	0·706443	0·706443	0·706443
0·602447	0·748231	0·748230	0·748230	0·748230	0·748230
0·651251	0·764455	0·764402	0·764403	0·764403	0·764403
0·672143	0·767725	0·767676	0·76776	0·767748	0·767750
0·6832	—	0·765720	0·76703	0·76707	0·767097
0·6908	—	—	0·7630	0·7660	—

TABLE 1. Comparison of results for wave speed squared when $\exp(-kQ/c) = 0{\cdot}5$, that is, wavelength/depth ≈ 9. — signifies no results obtainable. Results from Cokelet and Vanden-Broeck & Schwartz are the most accurate presented by them; as with the present method accurate results are often obtainable at lower levels of approximation.

The present method is a direct one in that values of stream function are obtained as a function of position. As the sharp-crested highest wave is approached the complex velocity potential w near the crest behaves like $z^{\frac{3}{2}}$ where z is the complex co-ordinate relative to the crest. Thus, the complex velocity $dw/dz \sim z^{\frac{1}{2}}$, also goes to zero at the crest. The expansion (5) does not explicitly include this local behaviour which, however, seems sufficiently smooth that this does not matter. If an inverse method were used, so that $z(w)$ had to be found, only half the unknowns need to be considered because the kinematic boundary condition could be satisfied exactly. However for the highest wave, near the crest, $z \sim w^{\frac{2}{3}}$, and the inverse of the complex velocity $dz/dw \sim w^{-\frac{1}{3}}$, showing singular behaviour which is more difficult to approximate numerically.

In an experiment to reduce the value of N required for higher and longer waves, the points on the free surface at which the equations were to be satisfied were clustered near the crest. This clustering is necessary in inverse numerical methods such as that used by Vanden-Broeck & Schwartz (1979), which otherwise spaces points coarsely near the crest. It was not expected to be an advantage in the present method, as the Fourier approximation requires a number of points on the long flat trough as well as in the vicinity of the crest. Results obtained did show that this was indeed the case, and it can be recommended that in all future applications, equi-spaced points be used.

A more detailed comparison between the present method and the results of Cokelet (1977) and Vanden-Broeck & Schwartz (1979) is presented in table 1. Results are given for the wave speed squared at various values of wave height, for a constant value of $\exp(-kQ/c) = 0{\cdot}5$, corresponding to a wavelength to water depth ratio of about 9, a moderately long wave. The Fourier method gave highly accurate results for waves up to about 99 % of the maximum height. It is interesting that although the coarsest approximation, $N = 16$, did not converge to a solution for the highest waves, it was still accurate for waves up to 97 % as high as the largest. For shorter waves the Fourier method is more accurate than for this case. At the other extreme of very long waves with a long flat region between crests, for which the Fourier approximation is not so well suited, results for wavelengths about 60 times the depth were obtained. For this

case $\exp(-kQ/c) = 0{\cdot}9$, direct comparison between the methods is rather more difficult because Cokelet did not present reliable results for the greatest wave heights and Vanden-Broeck & Schwartz presented no wave-height results. Values of kc^2/g obtained respectively by (i) the present method with $N = 64$, (ii) Cokelet and (iii) Vanden-Broeck & Schwartz are: maximum value, (i) 0·1654, (ii) 0·1645, (iii) 0·165038; and for the value corresponding to the highest wave solution obtained, (i) 0·1640, (ii) 0·162, (iii) 0·164437.

3. Fluid velocity under the crest: comparison with experiment

Theories based on expansions are not universally applicable to all situations since they depend on the expansion parameter being small and so do not produce good results for high waves. This is borne out when theoretical results are compared with those obtained by experiment.

Two experimental investigations which measure fluid velocities in a wave tank are those of Le Méhauté *et al.* (1968) and Iwagaki & Sakai (1970). The latter results seem to indicate the presence of a boundary layer which is much wider in the experimental situation than would be expected in the corresponding real situation and is not indicated in the profiles of Le Méhauté *et al.*; hence the variation with depth of the horizontal fluid velocity under the crest, as obtained by the method in § 2.2, is compared with the earlier results.

These experiments measured, in a finite time interval, the displacements of marked particles which moved with the fluid, that is, Lagrangian velocities were measured whereas the velocities determined in § 2 are Eulerian quantities. Although the instantaneous Lagrangian velocity is the same as the Eulerian velocity, the experimental values must be averaged over a finite time interval by virtue of the measurement process used, thereby underestimating the instantaneous velocity under the crest of the wave. It is possible to estimate the difference between the two, the mean velocity of a particle over a finite interval and the instantaneous velocity at a point; however, no indication of particle positions or time intervals was given and so no correction can be made in the subsequent comparison with Eulerian velocities.

Longuet-Higgins (1953) described the phenomenon of the steady particle drift caused by the viscosity of the fluid. For relatively large waves, as in the experiments considered here, the predicted effect is that a streaming of the particles should quickly be set up so that those near the bottom experience a relatively large steady drift velocity in the direction of wave propagation, while those near the top move in the reverse direction. However, there is no discernible evidence of this drift in either of the sets of experimental results and so, accordingly, no correction has been made to account for it.

Since the experiments were performed in a closed wave tank there is no net mass transport under the waves and the system of equations in § 2 must include (13*b*) with $c_s = 0$. Hence, the waves propagate with speed Q. Most previous comparisons with the experiments do not seem to have accounted for this.

Then, the horizontal Eulerian velocity under the crest at any instant is given by

$$u_c(y) = Q + B_0 + k \sum_{j=1}^{N} jB_j \frac{\cosh jky}{\cosh jkD}.$$

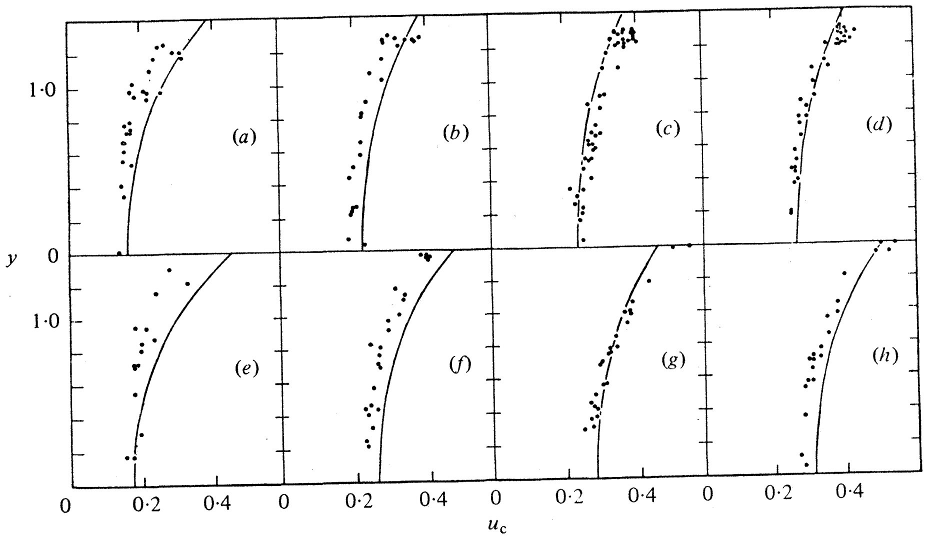

FIGURE 2. Horizontal fluid velocity under the wave crest, u_c, plotted against height y above the bottom: comparison between the present method (——) and the experimental results of Le Méhauté *et al.* (1968). (*a*) $H = 0{\cdot}434$, $\tau = 8{\cdot}59$; (*b*) $H = 0{\cdot}420$, $\tau = 15{\cdot}87$; (*c*) $H = 0{\cdot}389$, $\tau = 22{\cdot}49$; (*d*) $H = 0{\cdot}433$, $\tau = 27{\cdot}24$; (*e*) $H = 0{\cdot}499$, $\tau = 8{\cdot}59$; (*f*) $H = 0{\cdot}522$, $\tau = 15{\cdot}87$; (*g*) $H = 0{\cdot}492$, $\tau = 22{\cdot}49$; (*h*) $H = 0{\cdot}548$, $\tau = 27{\cdot}27$. (The horizontal scales vary slightly, the diagrams having been taken from results for dimensional velocities in water of varying depths.)

The velocity profiles for different values of wave height and wave period are shown in figure 2 along with the experimental points which were traced from a copy of Le Méhauté *et al.* (1968). For the first four cases, figures 2(*a*)–(*d*), with wave height approximately 0·4, agreement between this theory and experiment is quite close. Figures 2(*e*)–(*h*) show results for higher waves ($H \approx 0{\cdot}5$). Agreement is generally good except for case (*e*) which may reflect experimental difficulties because the theory gives better results for longer waves which are numerically more demanding. For each of the waves considered different runs were made with $N = 8$, 16 and 32. It was found that the value $N = 8$ was adequate, and the results for $N = 16$ and 32 were indistinguishable.

The agreement between the method presented here and experiment is far better than any of the other analytical theories as plotted by Le Méhauté *et al.*, even allowing for zero mass transport. Although the fifth-order cnoidal wave results of Fenton (1979), when adjusted to account for zero mean drift, compare well with the experimental results for the lower wave heights, that theory does not give such good results for the higher waves. Dean (1970*b*), using a similar method to that discussed in this paper, considered the same experimental cases but used $N = 5$ and with no correction for zero drift. As previously mentioned, this is adequate for the shorter waves, however for longer and higher waves it can give inaccurate results.

4. Application to shoaling of waves

4.1. *Introduction*

The problem of waves incident normally to a shoaling beach has traditionally been approximated by assuming that at any position the wave acts as if it were a steady wave on fluid of the local depth, assumed constant. Given this supposition, the simplest approach is to neglect frictional dissipation and assume that the wave period and energy flux remain constant in a transition from one depth to another. It is assumed that there is no reflexion of energy due to the changing depth, a valid approximation for small slopes (less than 4·5°, see Eagleson 1956).

This approach has been adopted by Eagleson, using linear wave theory and by Koh & Le Méhauté (1966) who used third-order and fifth-order Stokes theory to describe the waves. When compared with experiment, the higher-order theories do not give good results, because as the water becomes shallower, the waves become longer and higher and the Ursell criterion for Stokes waves $ak \ll (k\bar{\eta})^3$ is inevitably violated. In an attempt to produce results which are valid in the shallow region, Svendsen & Brink-Kjaer (1972) used first-order cnoidal theory matched to deeper water results produced by a Stokes approximation. This has the undesirable result of a discontinuity in wave height at the matching point, thereby indicating that one of or both the theories are in error at that depth and all subsequent depths. These methods can be, at best, only as good as the particular wave theory used.

One effect not accounted for by these methods is the relative depression of the local mean water level. Stiassnie & Peregrine (1980) assumed not only that wave period and wave-action flux (hence, energy flux) are conserved, but also that a Bernoulli constant and mass flux remain unchanged allowing for the wave-induced set-down of the water level. For waves in deeper water, they used the high-order Stokes expansion of Schwartz (1974) and Cokelet (1977) which gives accurate solutions for steadily-progressing waves. As the water becomes shallower and the waves become long, the Stokes expansion loses accuracy and so these waves were matched to the accurate solitary wave solution of Longuet-Higgins & Fenton (1974), making the assumption that long waves are accurately modelled by a train of solitary waves of finite length. This method gave good agreement with experiment, thus providing further justification for the use of local steady wave solutions in the study of wave shoaling. Interestingly, the wave set-down was found to have an insignificant effect.

While this approach can give results for gross integral quantities of the wave train, it is more difficult to obtain spatially-varying quantities such as fluid velocities which may be required in practical problems. The present method based on a Fourier approximation can be simply modified to provide a convenient and accurate means of modelling shoaling waves, giving all wave properties at each depth.

4.2. *Application of the present method*

The method of §§ 2.1, 2.2 is modified to include another variable F, the non-dimensional mean energy flux, as defined by (17*g*), and an additional equation relating this variable to the other quantities previously defined. Thus, using the results of § 2.3,

$$F = \tfrac{1}{2}c^3 - \tfrac{3}{2}c^2Q + c(2R - 1 - \tfrac{1}{2}QB_0 - \overline{\eta^2}) - Q(R-1), \tag{18}$$

where

$$\overline{\eta^2} = \frac{1}{2N}\left[\eta_0^2 + \eta_N^2 + 2\sum_{j=1}^{N-1}\eta_j^2\right].$$

It is usual in a shoaling problem to specify, in an initial depth of water $\bar{\eta}_0^*$, a wave of height H_0^* and period τ_0^*. The solution at this initial depth yields the energy flux according to equation (18). For succeeding depths, the quantities assumed to be conserved are the dimensional period $\tau^* = \tau_0^* = \tau(\bar{\eta}/g)^{\frac{1}{2}}$ and the dimensional energy flux $F^* = F\rho(g^3\bar{\eta}^5)^{\frac{1}{2}}$ while the wave height, H, is now considered to be a variable of the problem. Thus it is convenient to include yet another equation which specifies the wave height for the initial depth solution and the energy flux for each subsequent depth.

The additional equations for the Newton iteration method are

$$f_{2N+7} = \tfrac{1}{2}c^3 - \tfrac{3}{2}c^2Q + c(2R - 1 - \tfrac{1}{2}QB_0 - \overline{\eta^2}) - Q(R-1) - F = 0$$

and

$$f_{2N+8} = H - H_0^*/\bar{\eta}_0^* = 0 \quad \text{for the initial depth and}$$

$$f_{2N+8} = F - F_0 = 0 \quad \text{for the subsequent depths}$$

where F_0 is the energy flux non-dimensionalized according to the relevant mean depth.

The appropriate non-zero derivatives are

$$\frac{\partial f_{2N+7}}{\partial \eta_j} = \begin{cases} -c\eta_j/N, & j = 0, N, \\ -2c\eta_j/N, & j = 1, \ldots, N-1, \end{cases}$$

$$\frac{\partial f_{2N+7}}{\partial B_0} = -\tfrac{1}{2}cQ, \quad \frac{\partial f_{2N+7}}{\partial c} = \tfrac{3}{2}c^2 - 3cQ + 2R - 1 - \overline{\eta^2} - \tfrac{1}{2}QB_0,$$

$$\frac{\partial f_{2N+7}}{\partial Q} = -\tfrac{3}{2}c^2 - R + 1 - \tfrac{1}{2}cB_0, \quad \frac{\partial f_{2N+7}}{\partial R} = 2c - Q, \quad \frac{\partial f_{2N+7}}{\partial F} = -1,$$

$$\frac{\partial f_{2N+8}}{\partial H} = 1 \quad \text{for the first depth only, otherwise this derivative is zero.}$$

and

$$\frac{\partial f_{2N+8}}{\partial F} = 1 \quad \text{for depths subsequent to the first.}$$

Allowing H to be variable also necessitates the inclusion of

$$\frac{\partial f_{2N+4}}{\partial H} = -1.$$

The initial estimate for the energy flux is obtained, in accordance with the other variables, from a Stokes approximation:

$$F = \frac{\pi}{8}\frac{c^2H^2}{\tau}\frac{\sinh k \cosh k + k}{\sinh^2 k}.$$

This estimate is needed only for the first depth. Subsequently, for small changes in the depth, the previous solution may be used as a good initial approximation for the other variables provided that the change in depth is accounted for in the non-dimensionalization. For the results presented below, it was found that the following scheme, with the subscript 1 referring to the converged solution at the previous depth and 2 to the initial approximation at the next depth, provided a satisfactory estimate of the variables to be used in the first iteration at the new depth, using $r = \bar{\eta}_2^*/\bar{\eta}_1^*$, the ratio of the successive depths:

$$H_2 = H_1/r; \quad k_2 = k_1 r; \quad R_2 = 1 + (R_1 - 1)/r;$$

$$c_2 = c_1/r^{\frac{1}{2}}, \quad (\eta_j)_2 = 1 + \{(\eta_j)_1 - 1\}/r, \quad j = 0, \ldots, N;$$

$$(B_j)_2 = (B_j)_1/r^{\frac{1}{2}}, \quad j = 0, \ldots, N; \quad Q_2 = Q_1/r^{\frac{1}{2}};$$

with F_0 replaced by $F_0/r^{\frac{5}{2}}$ and τ replaced by $\tau/r^{\frac{1}{2}}$.

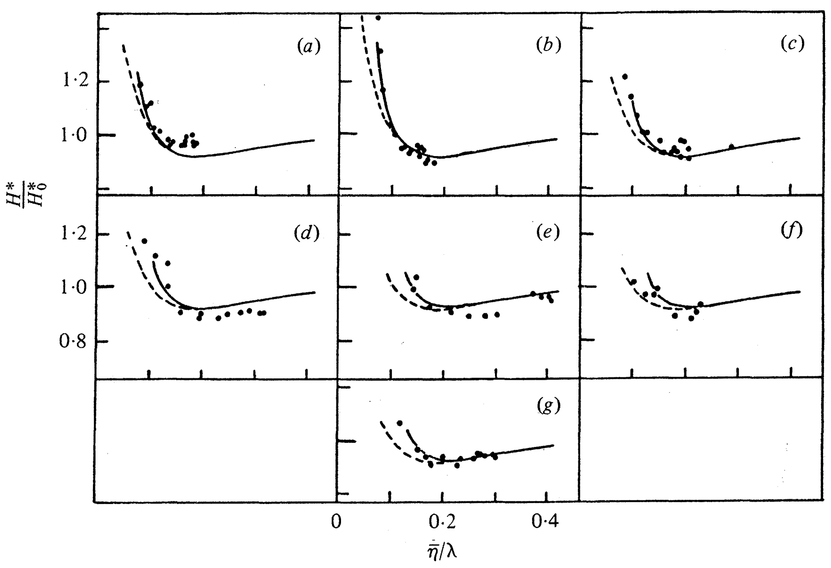

FIGURE 3. Wave shoaling: the ratio of local wave height to deep water wave height, H^*/H_0^*, as a function of the local dimensionless depth, $\bar{\eta}/\lambda$, as waves travel into the shallower water. The seven graphs are for the experimental cases of Eagleson (1956), values of period and height before shoaling are: (*a*) $H = 0{\cdot}106, \tau = 6{\cdot}13$; (*b*) $H = 0{\cdot}151, \tau = 7{\cdot}22$; (*c*) $H = 0{\cdot}202, \tau = 5{\cdot}96$; (*d*) $H = 0{\cdot}134, \tau = 4{\cdot}72$; (*e*) $H = 0{\cdot}131, \tau = 4{\cdot}02$; (*f*) $H = 0{\cdot}251, \tau = 5{\cdot}30$; (*g*) $H = 0{\cdot}204, \tau = 4{\cdot}74$. Together with the experimental points (.), the predictions of linear theory (– – – –) and present theory (——) are shown.

4.3. *Comparison with experiment*

The method described above was used to model two sets of wave-tank experiments. No mean drift could occur so that equation (13*b*) was used with $c_s = 0$.

Eagleson (1956) conducted experiments in a wave tank with a bed of uniform slope of about 1 in 15. He measured H^*, τ^* and $\bar{\eta}^*$ at one point before shoaling and used linear theory to predict the deep-water value H_0^*. His resultant points, from seven separate experiments are shown plotted on figure 3 which have been traced from a copy of his paper. The horizontal axis is $\bar{\eta}/\lambda$; the vertical axis is H^*/H_0^*, a dimensionless wave height. Also shown (dashed line) are the predictions of linear theory without requiring that mass transport be zero, as plotted by Eagleson. The results of the present method, shown by solid lines, agree well with experiment. At the start of shoaling, when the wave height is still small, linear theory is quite accurate but it diverges from the experimental results when the waves start to become large. The present method, however, seems to predict the wave height quite well.

Hansen & Svendsen (1979) conducted experiments on a uniform slope of 1 in 35. Some of their results have been used for comparison by Stiassne & Peregrine (1980) from whose paper the experimental points have been traced and are plotted on figure 4. Also shown (dashed line) in this figure 4(*b*) are three curves typical of the results obtained by Stiassnie & Peregrine. It seems that these curves were obtained by assuming different initial conditions so that each curve agrees with different parts of the experimental results. Each shows a non-uniqueness in H^*, consistent with the multi-valued nature of some wave properties when considered as a function of the

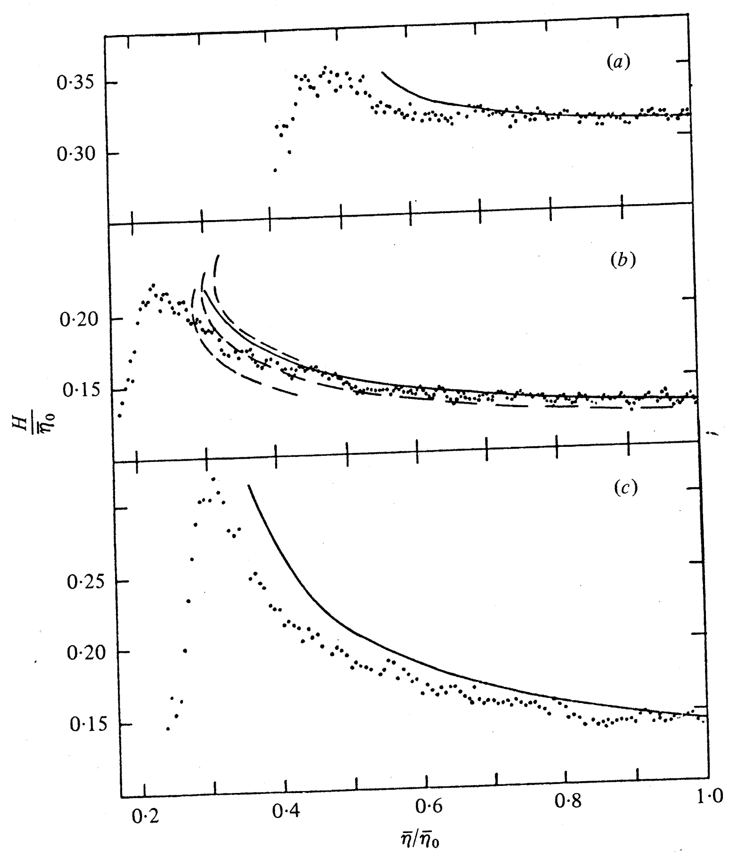

FIGURE 4. Wave shoaling: variation of wave height H relative to initial depth $\bar{\eta}_0$ as waves travel into shallower water, shown against the dimensionless local depth $\bar{\eta}/\bar{\eta}_0$. The three graphs are for experimental cases of Hansen & Svendsen (1979) whose experimental points are plotted with the predictions of the present method (——). Three curves of Stiassnie & Peregrine (1980) are shown (– – –) corresponding to differing initial conditions: (a) $H = 0{\cdot}31$, $\tau = 5{\cdot}72$; (b) $H = 0{\cdot}13$, $\tau = 0{\cdot}95$; (c) $H = 0{\cdot}14$, $\tau = 19{\cdot}04$.

wave height (see Longuet-Higgins & Fenton 1974). However, in the laboratory situation it does not seem that the upper limb would be attainable.

Results from the present theory are shown by the single solid curves, using as initial conditions the point at the right hand edge of each graph. It can be seen that agreement with experiment is quite good. The curves are congruent with those of Stiassnie & Peregrine except in the final stages before breaking where they continue the trend of the experimental results up to the breaking height but at a greater depth. This does not reflect any unusual virtue of the present method to describe *all* phases of shoaling, rather it is simply fortuitous that the lack of accuracy of the present Fourier approximation for the very highest and longest waves, in not accounting for the sharpness of the crest, causes the theoretical results to mimic the laboratory wave. All results in this section were produced from a Fourier approximation with $N = 16$.

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