Back to Papers

# Numerical comparisons of wave analysis methods

MURRAY TOWNSEND
JOHN D. FENTON

Department of Mechanical Engineering
Monash University, Clayton, Victoria 3168

**Summary:** Various wave analysis techniques are numerically compared using steady waves generated by a Fourier method. The sub-surface pressure data (such as would be acquired from a pressure transducer) is operated upon to derive water surface elevation data. More traditional global and more recent local methods are presented with the local methods providing greater accuracy over a wider range of wave and measurement conditions.

## 1. INTRODUCTION

An ongoing study in the Department of Mechanical Engineering at Monash University is investigating the measurement of waves using sub-surface pressure transducers. In this paper results will be presented from early numerical testing of various techniques using data generated from Fenton's [1] steady wave method.

The first method to be tested is the 'traditional' linear spectral method, utilising a transfer function (TF) derived from linear wave theory, which has been in use for many years and is known to have several important limitations. Debate on the accuracy (and validity) of this method has been drawn out and lively at times [2, 3]. More recently Kuo and Chiu [4] have suggested an empirical TF as a substitute for the linear one mentioned above.

A move to local time domain methods which act upon a short (meaning less than one wavelength) segment of the pressure data at each step has occurred since the beginning of the 1980's. Different techniques have been proposed by Daemrich [5], Nielsen [6, 7, 8], Fenton [9] and Fenton and Christian [10].

In this paper, the accuracy of the above methods (with the exception of Daemrich's [5]) will be tested throughout the region of possible water waves shown in Figure 1 (from Fenton [11]). Daemrich's method is not discussed as it was initially developed for manual analysis and Nielsen's initial investigations [6] are an extension of the concept anyway.

## 2. GLOBAL METHODS

### 2.1. Linear Spectral Method

The most common analysis technique used in this area is the linear spectral method, which has been presented and comprehensively discussed by Bishop and Donelan [2]. In a nutshell, the auto spectrum of the water surface elevation, $S_s(\omega)$ is related to the auto spectrum of the dynamic pressure at the pressure transducer, $S_p(\omega)$ by:

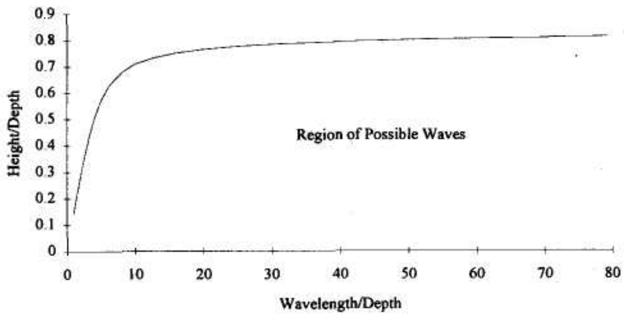


Figure 1: The region in which water waves are possible.

$$S_s(\omega) = \left[\frac{N(\omega)}{K_p(\omega)}\right]^2 S_p(\omega), \tag{1}$$

where $\omega$ is the angular frequency of each Fourier component, $N(\omega)$ is an empirical correction factor and $K_p(\omega)$ is the pressure response factor defined as:

$$K_p(\omega) = \frac{\cosh(k(\omega)y_p)}{\cosh(k(\omega)D)}, \tag{2}$$

where $D$ is the mean water level (MWL) and $y_p$ is the height of the pressure transducer from the sea bed.

The authors have, for the purpose of this paper set $N(\omega)$ to 1 for all $\omega$. Bishop and Donelan consider the presence of $N(\omega)$ as an attempt to compensate for poor measurements, instruments and/or analysis methods. As the authors are using 'exact' nonlinear waves and pressure traces to conduct these tests none of the above need be considered and only potential inadequacies in the linear spectral method will be highlighted.

It was necessary to determine a maximum $\omega$ above which the pressure response factor was not applied, as doing so would cause the method to 'blow-up' when $K_p(\omega)$ became small. To determine this limit, the ratio of the spectral amplitude to

$K_p(\omega)$ was determined at each $\omega$. When this ratio began to increase the method had started to 'blow-up'.

### 2.2. Empirical Transfer Function

The other global method to be tested is the empirical TF proposed by Kuo and Chiu [4]. In this method $N(\omega)$ is always set to 1 in (2) and the pressure response factor is replaced with an empirical TF:

$$K_e(\omega) = \exp\left(-0.905\frac{\omega^2(D-y_p)}{g} - 0.027\right) \qquad (3)$$

within the limits $0.1 \le \dfrac{\omega^2(D-y_p)}{g} \le 5.0; \dfrac{\lambda}{D} \le 14.29$

where $g$ is gravitational acceleration and $\lambda$ is the wavelength of each Fourier component.

This method is relatively new and untested. The authors have reservations about the range of applicability of the TF as published. The above limits were used to set the maximum $\omega$ above which $K_e$ was applied, not only to obey these limits, but also to avoid 'blow-ups' similar to those of the linear spectral method.

## 3. LOCAL METHODS

### 3.1. Local Approximation Methods

The first of the computer-based local methods to emerge were the two developed simultaneously by Nielsen [6, 7, 8]. The two are grouped under the umbrella of local approximation methods. A sine curve is passed through three points from the pressure data which are adjacent or with a small number of intermediate points between them. This sine curve is used to determine the 'local' frequency. The water surface elevation at the instant in time of the central point can then be calculated by one of two methods. One method applies a TF derived from stretched linear theory, the other applies a semi-empirical TF. The former TF, which the authors call Nielsen's first order method is expressed:

$$\hat{\eta}_n = \frac{p_n}{\rho g}\frac{\cosh k_n\left(D+\dfrac{p_n}{\rho g}\right)}{\cosh k_n z}, \qquad (4)$$

where $\hat{\eta}_n$ is the water surface elevation corresponding to the nth central pressure reading, $p_n$ is the nth central pressure reading, $k_n$ is the nth wave number derived from the local frequency calculated from the three pressure readings and $\rho$ is the water density.

The local frequency is determined by:

$$\hat{\omega}_n^2 = \frac{-p_{n-M} + 2p_n - p_{n+m}}{p_n(M\delta)^2}, \qquad (5)$$

which is an estimate corrected by:

$$\omega_n^2 = \hat{\omega}_n^2\left[1 + \frac{1}{12}(\hat{\omega}\delta)^2\right]. \qquad (6)$$

The latter TF Nielsen derived from Dean [12]:

$$\hat{\eta}_n = \frac{p_n}{\rho g}\left[A\left(\frac{y_p}{D}\right)\frac{-p_{n-M} + 2p_n - p_{n+M}}{p_n g(M\delta)^2}\left(D + \frac{p_n}{\rho g} - y_p\right)\right] \qquad (7)$$

where $M \approx \sqrt{D/g}/\delta$ and is a multiplier used to filter noisy data, $\delta$ is the sampling period of the data, and $A(y_p/D) = 0.67 + 0.34y_p/D$ and accounts for the height of the pressure transducer above the sea bed.

Both the above are extremely simple to apply with little computational effort required. In this study $M$ was set to 1 as no noise was present in the input and the authors felt that this gave a better indication of the method's robustness. The exception to this was when simulated noise was added to the pressure signal. In this case $M$ was calculated by the above equation.

### 3.2. Local Polynomial Approximations

The other two local methods are called local polynomial approximation (LPA) techniques and were developed by Fenton [9] and Fenton and Christian [10]. Both utilise the principle of low-degree polynomial approximation, partly based on least-squares approximation methods and partly on solving locally the full nonlinear equations of motion. The first approach was to approximate the complex velocity potential as follows:

$$\begin{aligned} w(x,y,t) &= \phi(x-ct,y) + i\psi(x-ct,y) \\ &= \sum_{j=0}^{J}\frac{a_j}{j+1}(z-ct)^{j+1}, \end{aligned} \qquad (8)$$

where $z=x+iy$, and the surface elevation is given by

$$\eta(x,t) = \sum_{j=0}^{J} b_j(x-ct)^j. \qquad (9)$$

Expansion (6) satisfies Laplace's equation identically throughout the flow and the bottom boundary condition ($v(x,0,t)=0$) is satisfied if the coefficients $a_j$ and $b_j$ are real.

To satisfy the necessary boundary conditions on the free surface the steady kinematic equation is invoked such that:

$$\psi(x-ct, \eta(x-ct)) = -Q, \qquad (10)$$

where Q is a constant, and the steady Bernoulli equation:

$$R = \frac{1}{2}\left|\frac{dw}{d(z-ct)}\right|_s^2 + \eta, \tag{11}$$

where $R$ is the Bernoulli constant and $s$ denotes the surface $y=\eta$.

Bernoulli's equation is also written about the position $(0, y_p)$, the position of the pressure transducer, expressed as a Taylor series in $x$-$ct$:

$$p(x, y_p, t) = R - \frac{1}{2}\left|\frac{dw}{d(z-ct)}\right|_{y_p}^2 - y_p = \sum_{j=0}^{J} p_j (x-ct)^j \,. \tag{12}$$

The $p_j$ are calculated using a least squares fit across $K$ data points where $K$ is an odd integer with $(K$-1)/2 data points each side of the point of interest. For the above method $K$=21 was found to give good results for both smooth and noisy data and is the value used in these tests.

By manipulation of (8), (9), (10), (11) and (12) and isolating powers of ($x$-$ct$), a system of nonlinear equations in terms of the unknown $a_j$ and $b_j$ are obtained. The solution of these equations is performed for each point in the pressure series using direct iteration to achieve convergence. The surface elevation data obtained was then passed though a simple 3-point smoothing routine. Space does not permit explanation of the details regarding the solution of these equations for the $a_j$ and $b_j$ coefficients for this full nonlinear LPA (referred to as Full LPA in the following text).

The second local polynomial method (Simple LPA) [10] is somewhat simpler in that a point value is used to describe each value of $\eta$ at a corresponding $t_n$ as opposed to the polynomial expansion in the former method [9]. The resulting solution is much simplified and it is claimed [10] that accuracy is as good as that of the full local polynomial solution with $K$=17.

For both methods little is to be gained by using higher than 4th order polynomials ($J$=4) as accuracy does not improve greatly and the resulting nonlinear equations become extremely cumbersome to manipulate.

## 4. TEST CONDITIONS

As previously stated, pressure traces for nonlinear waves have been generated using a Fourier approximation method for accurately solving steadily progressing wave problems. These pressure traces were used as input data for the above methods and the output of computed water surface elevation was compared to the corresponding solution from the program for accuracy.

Wave data was generated across a rectangular grid within the region of possible waves in Figure 1. Intervals of 0.1 on the horizontal axis ($\log_{10}$ ($\lambda/D$)) and the vertical axis ($H/D$) were used, resulting in a total of 107 points in that region. In addition six different values of $y_p/D$ were tested, from 0 to 0.5 in steps of 0.1. Only a small selection of the total output is shown here. There are 64 points per wave length with a further 5 points added to each end of the test wave.

## 5. RESULTS

Figures 2 to 7 display results from the methods mentioned above for three different $\lambda/D$ ratios. $\lambda/D$ ratios of 6.31, 10 and 19.95 at an $H/D$ ratio of 0.6 and $y_p/D$ of 0 are presented. In addition for $\lambda/D$=10 at $y_p/D$=0 a pressure signal contaminated with simulated electronic noise is shown along with uncontaminated data at $y_p/D$ =0.5. The legend shown in Figure 2 is valid for figures 2 to 7. Additional results are described in the captions for the figure in which they appear.

All figures but Figure 4 show results for all methods except Kuo & Chiu's empirical spectral method. This was excluded as accuracy was extremely poor and nothing was to be gained by including it. This was found to be the case for all $y_p/D$=0 tests, even when the test waves were within the limits imposed upon the method ($\lambda/D$=6.31,10).

Figure 2 shows results for a $\lambda/D$ of 6.31 , $H/D$ of 0.6 and $y_p/D$=0. This $\lambda/D$ ratio seemed to be a lower limit for reasonable results in general. Nielsen's two methods lie almost on top of one another and provide an accurate estimate of the $H/D$ ratio although the calculated shape of the profile is quite poor. Nielsen's methods provide a remarkably accurate estimates of the wave crest and trough but elsewhere are prone to inaccuracy. Neither the linear spectral or simple LPA methods provide very good results for such a short wave. The full LPA provides an accurate estimate of the profile but underestimates the crest.

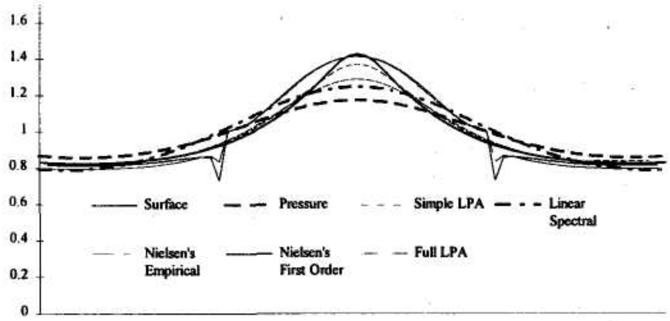


Figure 2: $\lambda/D$ = 6.31, $H/D$ = 0.6, $y_p/D$ = 0.

Figure 3 has a $\lambda/D$ ratio of 10 with the same $H/D$ and $y_p/D$ values as Figure 2. Nielsen's methods behave in a similar manner to previous results and the linear spectral method has provided a good estimate of the crest but an extremely poor estimate of the trough. This occurs because of the large number of frequency components needed to adequately describe both a sharp crest and a long, flat trough. The linear TF becomes so small that the method blows up before the trough can be smoothed. The simple LPA method provides a generally good estimate of the wave profile but falls short of the crest, with the full LPA providing a good fit overall.

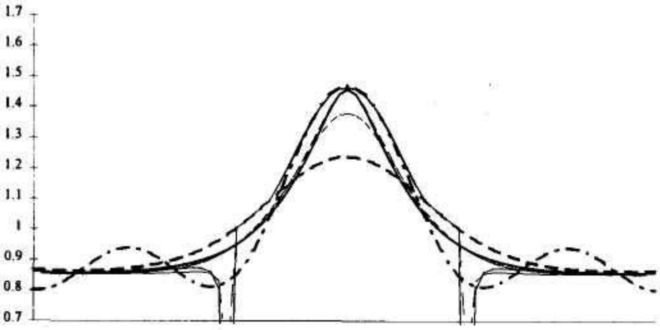


Figure 3: $\lambda/D$ = 10.0, $H/D$ = 0.6, $y_p/D$ = 0.

An improvement in accuracy for the LPA, linear spectral and Kuo & Chiu's methods occurs when the same wave as Figure 3 is measured further up the water column, at $y_p/D$=0.5 (Figure 4). The LPA's are extremely accurate in both profile and height estimates, while all other methods overestimate the crest slightly. Both spectral methods fail to adequately describe the trough, the manner of their failure has been discussed in the previous paragraph.

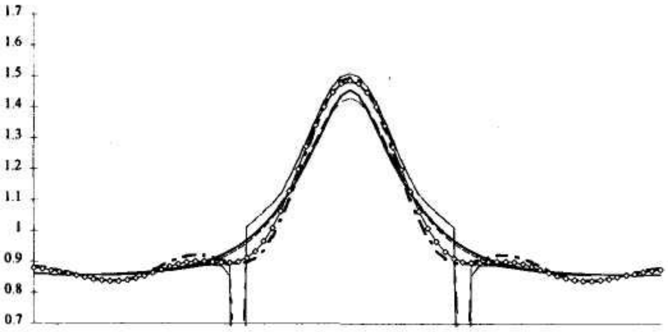


Figure 4: $\lambda/D$ = 10.0, $H/D$ = 0.6, $y_p/D$ = 0.5. Kuo & Chiu's method (—◇—).

Figure 5 is again the same wave as in Figure 3 ($y_p/D$=0) but with simulated random noise added to the pressure signal. To do this a random number generation routine was used to cause a fluctuation about the true pressure reading with a maximum error $H_n$ of ± $0.025H_p$, where $H_p$ is the height of the pressure signal from trough to crest. Very little change in the accuracy of the simple LPA occurs but the linear spectral method fails badly and both Nielsen's methods (for $M$=6) amplify the noise while maintaining their overall positions. Nielsen's first order method with $M$ set to 1 is included to show the considerable smoothing effect of the multiplier $M$.

The addition of noise highlighted the fragility of the full LPA. The authors originally used 17 points per computational panel but the method failed to converge when dealing with the noisy data. Several different values of $K$ were then tested with failures occurring with several other noise free data sets. This indicated that the full LPA method does not always converge for all values of $K$ and furthermore that these occurrences may be case dependent. This would be a major drawback for this method. For 21 points or more the method was reliable and accurate.

Figure 6 is a somewhat longer wave than previously displayed, at $\lambda/D$=19.95. However the behaviour of the methods shown is similar to that of Figure 3, with the most obvious difference being Nielsen's methods' tendency to slightly overestimate the crest. Kuo & Chiu's method is no longer valid at this $\lambda/D$ ratio, the limit being $\lambda/D \leq 14.29$.

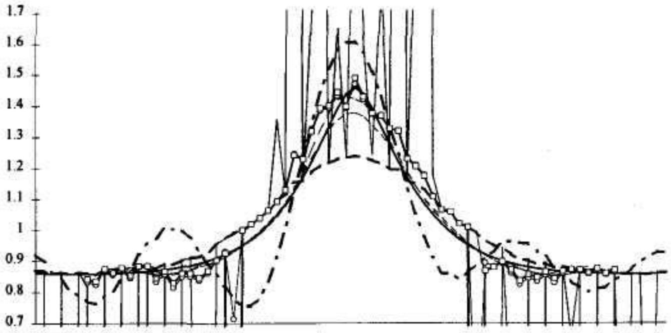


Figure 5: $\lambda/D$ = 10.0, $H/D$ = 0.6, $y_p/D$ = 0. Random noise added to pressure signal, $H_n = 0.05H_p$. Nielsen' empirical, $M$=6 (—○—) and Nielsen's first order, $M$=6 (—□—).

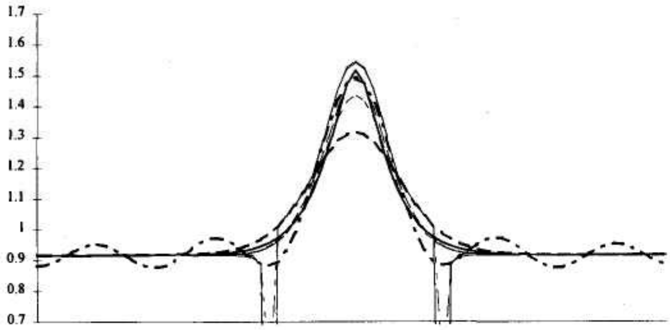


Figure 6: $\lambda/D$ = 19.95, $H/D$ = 0.6, $y_p/D$ = 0.

All the above waves are extreme examples of steady waves at $H/D$=0.6. A more commonly encountered example may be an $H/D$ =0.3 such as in Figure 7. In this case $\lambda/D$=10 and $y_p/D$=0. All methods barring Kuo & Chiu's behave quite well with Nielsen's empirical method overestimating the crest slightly and the LPA's underestimating. The linear spectral method estimates the wave height well but the profile is poorer than the other methods.

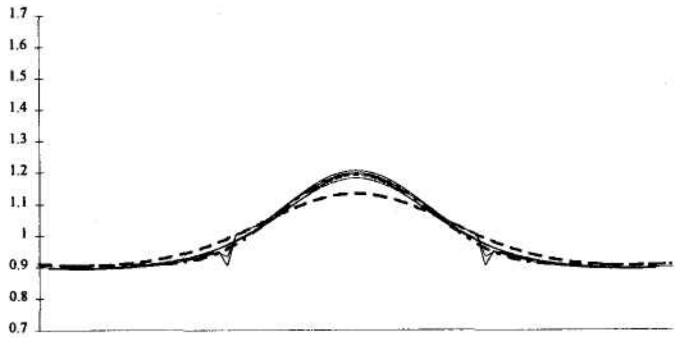


Figure 7: $\lambda/D$ = 10.0, $H/D$ = 0.3, $y_p/D$ = 0.

## 6. CONCLUSIONS

A comprehensive numerical comparison of a variety of global and local wave analysis methods has been conducted. The aim was to compare the accuracy of these methods for determining wave surface elevation data from sub surface pressure readings.

Overall the local methods are more effective in dealing with highly nonlinear steady waves than are the global methods. Unsurprisingly, all methods provided better results with higher $y_p/D$ ratios.

Kuo & Chiu's empirical spectral TF behaved extremely poorly when the pressure data was taken at the bottom of the water column and only gave what the authors consider to be acceptable results when $y_p/D$=0.5. In addition, this method is claimed to only be valid at or below $\lambda/D = 14.29$ which limits its range of applicability compared to the others tested.

The linear spectral method performed better for moderate waves but behaves poorly when the waves are steep and $y_p/D$ is small. Of the global methods tested, the linear spectral method provided more consistent and accurate results than Kuo & Chiu's method.

Both Nielsen's local methods behaved similarly, generally predicting wave heights well but not estimating the wave profile accurately. Nielsen's methods did not deal very well with the noise added to the pressure signal although a post analysis smoothing routine would probably result in fairly smooth curves. Of the two, Nielsen's empirical method is the simpler to apply, without sacrificing any accuracy compared to the first order method. However a disadvantage of Nielsen's methods is the total failure just above the trough or the 'shoulder' of the wave. The inability of these methods to determine a local frequency in these areas is a significant problem.

The simple LPA method does not perform as well as Nielsen's methods when predicting wave heights from pressure readings at low $y_p/D$ values but generally describe the wave profile with more accuracy than all other methods (with the exception of the full LPA) and is extremely accurate when $y_p/D$ is high. It is able to deal with noise in a satisfactory manner, better in fact than the full LPA method. In addition to this, the simple LPA is much easier to derive and program than the full LPA.

The apparent fragility of the full LPA method is unfortunate as its accuracy is higher than all other methods if no noise is present. This fault and the necessity for some form of smoothing to be performed on the output poses a real problem for the implementation of this method.

This paper has not investigated these methods when used with experimental data. A programme of experimental testing is the next stage in this ongoing study, which will include both periodic and irregular sea states. Surface elevation, pressure and fluid velocity will be measured simultaneously and the applicability of the various methods will be tested.

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