A ONE-DIMENSIONAL FLOW MODEL FOR FLOW OVER TRAPEZOIDAL PROFILE WEIRS

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ABSTRACT

This paper presents a simple one-dimensional flow model for simulating steady transcritical flow over short- and broad-crested trapezoidal profile weirs as well as for the development of discharge rating curves for these weirs under free flow condition. The proposed model is developed to allow for curvature of the free surface and a non-hydrostatic pressure distribution. The finite difference method is employed to discretise and solve the resulting nonlinear flow equation. A comparison of the computed results with the corresponding experimental data for smooth and rough beds flow conditions is presented. Good agreements for flow surface profiles and rating curves were obtained from this comparison. The model predicts accurately the transcritical flow surface profiles regardless of the curvature of the streamlines, demonstrating the capability of the model for handling flow situations with predominant non-hydrostatic pressure distribution effects. The model results also indicate the detailed dependence of the global flow characteristics of these weirs on the curvature of the streamlines.

1.   INTRODUCTION

Numerical and experimental studies of flow over trapezoidal profile weirs have a number of applications especially in the analyses of flood flows over common types of civil engineering structures such as highway and railway embankments. The transcritical flow characteristics of these weirs often provide important boundary conditions for the application of large two-dimensional flow models. Also, the hydraulic advantage of higher discharge capacity of the trapezoidal profile weirs compared to broad-crested weirs with vertical faces makes them very attractive in practice as a discharge-measuring device. Thus, the development of head-discharge relationships of such weirs assumes considerable practical importance. The significant nature of the flow over these weirs, especially in the vicinity of flow transition from subcritical to supercritical state, is a strong departure from the hydrostatic distribution of pressure caused by sharp curvatures of streamlines. The common computational flow models, which assume uniform velocity and hydrostatic pressure distributions, do not retain accuracy for such types of flow situations. The two-dimensional nature of this flow problem requires relatively accurate methods for exact simulation of the flow situation. In

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this study a model based on a higher-order Boussinesq-type equation will be employed for the simulation of such type of flow problem numerically.

A review of the literature shows that little effort has so far been made to study the numerical modelling of flow over trapezoidal profile weirs particularly for the establishment of rating curves. Contrary to this, the problem of flow over such weirs has been extensively studied experimentally. Most of the experimental works were performed towards the understanding of the local flow characteristics of these weirs and also the determination of the coefficients of discharge under free and submerged flow conditions (see e.g., Kindsvater (1964), Fritz and Hager (1998)). A simple numerical procedure was applied to develop rating curves for flow over broad-crested weirs by incorporating directly the discharge coefficients. The procedure was developed based on the lowest-order energy equation which assumes uniform flow at the gauging station and control section, and constant head between this station and section (Bos (1985)). This rating procedure provides solution to irrotational flow problems with negligible curvature of streamlines since its application is limited by the ratio of the total energy head, H_0, to weir crest length, L_w ($H_0 / L_w \leq 0.50$). However, the procedure is inappropriate to develop a rating curve for flow over short-crested weirs where the effects of non-hydrostatic pressure and nonuniform velocity distributions are significant. More recently Collins and Catalano (2001) studied the ability of the DELFT-FLS model to predict accurately the crest-referenced head of a broad-crested weir. The study aimed to explore the capabilities and limitations of the model. The discharge rating curves of the weir were simulated for free flow condition and the predicted results were compared with the results of the common broad-crested weir formula for the range in which this equation estimates the discharge accurately ($0.08 \leq H_0 / L_w \leq 0.33$). However, the study did not include flow simulations over short-crested weirs ($0.33 < H_0 / L_w \leq 1.50$) where the curvature of the streamlines above the weir crest has a considerable influence on the head-discharge relationships. This review demonstrates that a general model, which includes the effects of the curvature of the streamlines implicitly or explicitly, is necessary to provide head-discharge relationships for short- and broad-crested weirs. In this work a one-dimensional model will be employed for flow simulation over trapezoidal profile weirs with smooth transition curves introduced at the four corners of the profile for the purpose of integrating the flow equation continuously at these points.

Therefore the main objectives of this paper are: i) to model transcritical flow over trapezoidal profile weirs with smooth and rough beds using a Boussinesq-type equation numerically for the purpose of developing discharge rating curves; ii) to examine the influence of the curvature of the streamlines on the model solutions; iii) to demonstrate the validity of the model by a number of laboratory experiments.

2.   FORMULATION AND SOLUTION METHOD

2.1   Governing Equation

For steady flow in a rectangular channel, the Boussinesq-type momentum equation developed by Fenton (1996) reads as

$$\frac{\beta q^2}{4}\frac{d^3 H}{dx^3} + \frac{\beta Z_b' q^2}{2H}\frac{d^2 H}{dx^2} + \left(1 + Z_b'^2\right)\left(\left(gH - \beta\frac{q^2}{H^2}\right)\frac{dH}{dx} + gH\left(Z_b' + S_f\right)\right) + \omega_0 \beta q^2\left(\frac{Z_b'''}{2} + \frac{Z_b' Z_b''}{H}\right) = 0, \quad (1)$$

in which H is the depth of flow; Z_b', Z_b'' and Z_b''' are the first, second and third derivatives of the bed profile respectively; S_f denotes the friction slope, calculated from the Manning equation or

smooth boundary resistance law; q is the discharge per unit width; β refers to the Boussinesq coefficient; g is gravitational acceleration; and ω_0 is a constant factor to reflect the effect of the bed in determining the elevation of the surface and the associated dynamic pressures due to slow moving flow near the bottom of the flow boundary. Fenton (1996) suggested a value of slightly less than one for ω_0. In the formulation of this equation, the curvature at the surface is approximated by $\kappa_H \cong d^2 H / dx^2 + Z_b^{"}$ and at the bed by $\kappa_b \cong Z_b^{"}$.

This equation implicitly includes the effect of the vertical acceleration to model two-dimensional flow problems where more vertical details are significant and essential. For the case of weakly curved free surface flow with negligible curvature of streamlines in a constant slope channel, the flow surface and bed curvatures terms vanish to zero. Under this flow condition, the above equation reduces to the gradually varied flow equation. The above equation, eq. 1, will be used in this study to simulate transcritical flow over trapezoidal profile weirs for the purpose of developing discharge rating curves. These weir profiles are characterized by the presence of discontinuous bed geometry properties at the four corner points. Smooth transition curves should be fitted at these points in order to integrate eq. 1 continuously within the solution domain. A simple curve described by a polynomial function of degree five was used. The length of this curve was fixed based on the criteria that the error associated with the prediction of the bed profile elevation around the discontinuous point is less than 3%.

2.2   Problem Formulation and Boundary Conditions

The computational domain for the numerical solution of the weir flow problem is shown in Figure1. In this figure AB is the inflow section, CD is the outflow section and MUVN is the trapezoidal profile weir. The inflow section of the computational domain is located in a region where the flow is assumed to be nearly horizontal, with uniform velocity and hydrostatic pressure distributions. This quasi-uniform flow condition before the inflow section of the solution domain simplifies the evaluation of the boundary values at this section using the gradually varied flow equation, eq. 2. Thus, for a given depth at the inflow section the slope of the water surface, S_H, can be evaluated from the following equation:

$$S_H = \frac{dH}{dx} = \frac{S_0 - S_f}{1 - \beta Fr^2}, \tag{2}$$

in which Fr is the Froude number; and S_0 is the bed slope. For the subcritical flow upstream of the weir, the Froude number squared is sufficiently small and can be neglected. Using this approximation, the curvature of the flow surface, κ_H, at the inflow section after differentiating eq. 2 with respect to x becomes

$$\kappa_H = \frac{-2q^2 B^2}{K^3} \frac{dK}{dH} S_H, \tag{3}$$

where K is the conveyance factor, can be determined from the Manning equation; B is the width of the channel. Similarly, the gauging station GS should be situated sufficiently far upstream of the weir to avoid the influence of the curvature of the water surface on the magnitude of the estimated overflow depth. According to Bos et al. (1984) this section is located at a distance of the larger of the following two values: i) between two and three times the maximum crest referenced head from the upstream edge of the weir crest; ii) the maximum crest referenced head from the heel of the

trapezoidal profile weir. From the computational point of view, however, the maximum overflow head is not known a priori to fix the position of the gauging station. In the present study, the overflow head corresponding to the maximum discharge at the heel of the trapezoidal profile weir will be used to locate the gauging station approximately.

For the given flow depth, AB, and discharge at the inflow section, it is required to determine the flow surface profile, AC, and the corresponding overflow head at the gauging station GS. For this purpose, the computational domain bounded by the free surface, inflow and outflow sections, and the solid flow boundary is discretized into equal size steps in x as shown in Figure 1.
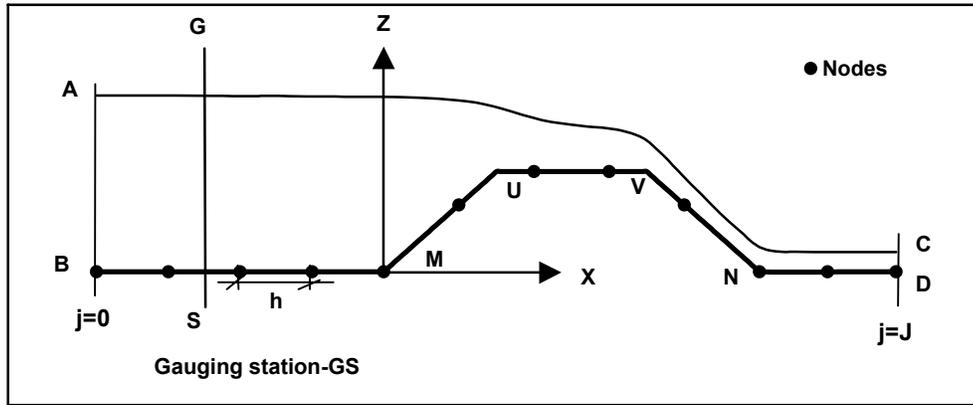
Figure 1 Computational domain for transcritical flow over weir.

2.3   Numerical Modelling and Solution Procedure

A numerical solution is necessary since closed-form solution is not available for this nonlinear differential equation. The finite difference approximations are used in this work to discretise the governing equation. This formulation is very simple to code and extensively used to solve linear or nonlinear differential equations. For the purpose of discretization, eq. 1 can be represented by a simple general equation as

$$\frac{dH^3}{dx^3} + \xi_0 \frac{dH^2}{dx^2} + \xi_1 \frac{dH}{dx} + \xi_2 = 0, \tag{4}$$

where ξ_0, ξ_1 and ξ_2 are the nonlinear coefficients associated with the equation. Higher-order finite difference approximations are employed here to replace the derivative terms in this third-order differential equation in order to reduce the truncation errors introduced in the formulation due to the finite difference quotients (see e.g., Fletcher (1991)).

The upwind finite-difference approximations (Bickley (1941)) for derivatives at node j in terms of the nodal values at $j-3$, $j-2$, $j-1$, j and $j+1$ are introduced into eq. 4 for the purpose of discretising the equation. After simplifying the resulting expression and assembling similar terms together, the equivalent finite difference equation reads as

$$H_{j-3}\left(12 - 2\xi_{0,j}h - 2\xi_{1,j}h^2\right) + H_{j-2}\left(-72 + 8\xi_{0,j}h + 12\xi_{1,j}h^2\right) + H_{j-1}\left(144 + 12\xi_{0,j}h - 36\xi_{1,j}h^2\right) +$$

$$H_j\left(-120 - 40\xi_{0,j}h + 20\xi_{1,j}h^2\right) + H_{j+1}\left(36 + 22\xi_{0,j}h + 6\xi_{1,j}h^2\right) + 24\xi_{2,j}h^3 = 0, \tag{5}$$

where h is the size of the step. In the solution domain, eq. 5 is applied to evaluate nodal values at different points. However, the use of eq. 5 at $j=1$ and $j=J$ will introduce unknowns external to the computational domain. Using the backward difference approximations in terms of nodal points $j,\ j-1,\ j-2\ldots$ for the derivative terms in eq. 4, the finite difference equation at the outflow section after simplifying the resulting expression becomes

$$H_{j-4}\left(36+22\xi_{0,j}h+6\xi_{1,j}h^2\right)+H_{j-3}\left(-168-112\xi_{0,j}h-32\xi_{1,j}h^2\right)+H_{j-2}\left(-288+228\xi_{0,j}h+72\xi_{1,j}h^2\right)+$$

$$H_{j-1}\left(-216-208\xi_{0,j}h-96\xi_{1,j}h^2\right)+H_{j}\left(60+70\xi_{0,j}h+50\xi_{1,j}h^2\right)+24\xi_{2,j}h^3=0. \tag{6}$$

Since the value of the nodal point at $j=0$ is known, the values of the imaginary nodes at $j=-1$ and $j=-2$ can be determined from the estimated water surface slope, S_H, and curvature of the free surface, κ_H, at the inflow section. Using similar discretization equations as above for the water surface slope and curvature at inflow section and the expanded form of eq. 5 at $j=0$, the explicit expressions for the nodal value at $j=-1$ and $j=-2$ are

$$H_{-2}=4H_1-15H_0+12H_{-1}+6hS_H-6h^2\kappa_H, \tag{7}$$

$$H_{-1}=\left(\frac{1}{\Phi-\Omega}\right)\left(\frac{A\left(27H_1-80H_0+24hS_H-36h^2\kappa_H\right)+B\left(6hS_H-15H_0+4H_1-6h^2\kappa_H\right)+}{H_1E+H_0\Gamma+Y}\right), \tag{8}$$

where:

$$\Gamma=12h^2\xi_{1,0}+8h\xi_{0,0}-72;\quad E=-2h^2\xi_{1,0}-2h\xi_{0,0}+12;$$

$$\Omega=36h^2\xi_{1,0}+12h\xi_{0,0}+144;\quad \Phi=-54A-12B;\quad A=6h^2\xi_{1,0}+22h\xi_{0,0}+36;$$

$$B=20h^2\xi_{1,0}-40h\xi_{0,0}-120;\ Y=24h^3\xi_{2,0}.$$

The solution of the nonlinear flow equation based on a boundary value technique requires an initial estimate of the position of the free surface profile. This makes the solution of the flow problems relatively more difficult due to the fact that the location of the free surface profile is not known a priori. Generally, such problems must be solved by iterative methods, which proceed from an assumed initial free surface position. Convergence of the iteration procedure to a final profile that satisfies the boundary conditions may be dependent to some extent on the choice of the initial flow surface profile. In this work the Bernoulli and continuity equations are employed to obtain the initial flow surface profile for commencing the iteration solution. To simulate the flow surface profile, eqs. 5 and 6 are applied at different nodal points within the solution domain and this results a banded system of nonlinear algebraic equations. These equations together with eqs. 7 and 8, and a boundary value at the inflow section, are solved by the Newton-Raphson iterative method with a numerical Jacobian matrix. The convergence of the solution is assessed using the following criterion:

$$\sum_{j=1}^{m}\left|\delta H_j\right|\leq tolerance,$$

where δH_j is the correction depth to the solution of the nodal point j at any stage in the iteration; m is the total number of nodes in the solution domain excluding nodes having known values. In this

study, a tolerance of 10^{-6} is used for the convergence of the numerical solution. The overflow head at the gauging station can be determined from the predicted flow depth at this station and a known height of the trapezoidal profile weir above the upstream floor level.

3.   EXPERIMENTAL SET-UP

The experiments were performed in a horizontal flume 7100 mm long, 380 mm high, and 300 mm wide. The flume and the trapezoidal profile weirs were made of Plexiglass. The water was supplied to the head tank from a sump through 115 mm diameter pipe with a valve for controlling the discharge. Various flow improving elements were provided upstream of the trapezoidal profile weirs to obtain a smooth flow without large-scale turbulence. Symmetrical trapezoidal profile weirs of 150 mm high, crest lengths 100 mm, 150 mm and 400 mm respectively and side slope 1V: 2H were tested at different discharges. For the purpose of assessing the performance of the model on rough flow boundary, the surface of the 150 mm weir model including the bed of the flume was roughened using pieces of mild steel wire screen with mesh size 6.5 mm square. The diameter of the wire from which the screen was made was 0.56 mm. This method of roughening has been used in the past for simulating bed roughness in free surface flow (see e.g., Kindsvater (1964)). The roughened model was also tested at different discharges.

A volumetric tank system was used to measure the discharge. The system consists of a tank with plan dimensions of 200 cm by 150 cm and depth of 450 cm, and an inclined manometer (56.62 ° to the horizontal) for water level measurement in the tank. Tank filling time longer than one minute was used to minimize errors associated with the starting and stopping of the stopwatch. The longitudinal flow surface profile was observed with a manual point gauge of reading accuracy 0.10 mm. For recording the bed pressure, steel pressure taps of external diameter 3 mm were fixed along the centreline of the weir model with maximum horizontal spacing of 80 mm, but the spacing was much closer near to the edges of the weir crest. These pressure taps were connected to vertical water piezometers of reading accuracy to 1 mm by long plastic tubes. For each experiment, the base reading for the pressure taps was obtained immediately after the flow was shut off. The velocity distributions of the flow at different sections were measured using an Acoustic Doppler Velocimeter (ADV) with a two-dimensional side-looking probe. According to the criteria given by Ranga Raju et al. (1990), the effects of viscosity and surface tension on the experimental results were negligible.

4.   MODEL RESULTS

The numerical model described in the previous section is used to simulate steady flow over short- and broad-crested trapezoidal profile weirs for transcritical flow situation. In order to assess the effect of the streamline curvature on the solution of the model, free flow situations with different H_0 / L_w values were considered. The magnitude of this ratio determines the degree of the curvature of the flow over the crest of the weir. For $H_0 / L_w \leq 0.50,$ the curvature of the streamline over the crest is insignificant except near to the edges of the weir crest (see e.g., Bos (1978)). Consequently, the expected effects of the non-hydrostatic and/or nonuniform velocity distributions on the behaviour of the flow might be insignificant. For computational simplicity β is taken as unity in the above flow equation. In all cases of flow simulations, the size of the steps is designed to be fine enough to meet the requirements of reasonable accuracy. All computational results presented here were independent of the effect of spatial step size.

4.1 Model Results for Smooth Bed

Since this part of the experiment was performed in a laboratory flume, a smooth boundary resistance law was applied to estimate the friction slope for the numerical model. The comparison of the computed and measured free surface profiles for transcritical flow over short- and broad-crested trapezoidal profile weirs is shown in Figure 2. A good agreement is seen between the observed and predicted free surface profiles in both the subcritical and supercritical flow regions, and for all cases of curvatures of the flow surface over the crest of the weirs. The comparison results indicate that the model simulates the weir flow problem accurately irrespective of the degree of curvature of the streamlines.

The computed head-discharge curves for flow over short- and broad-crested trapezoidal profile weirs are compared with the experimental results in Figure 3. The numerical solutions demonstrate good agreement with the experimental data for both weirs. This figure also compares the simulated discharge rating curves of the short- and broad-crested types of these weirs. Depending on the magnitude of H_0 / L_w, the same type weir can act as a broad-crested or a short-crested weir. At low flow rates the rating curves for the short- and broad-crested type of trapezoidal profile weirs are identical; indicating the insignificance of the influence of the curvature of the streamlines on the discharge characteristics of these weirs at this level of the flow rates (see Figure 3). As the discharge increases, clear differences between the head-discharge curves of the broad-crested and short-crested weirs are observed. This difference is due to the large increase of curvature of the streamlines of the flow over the crest of the short-crested trapezoidal profile weir. It can be observed from Figure 3 that in the region of relatively high flow rate, the overflow depth required to pass the given discharge over the broad-crested weir is larger than the corresponding overflow depth for the short-crested weir. This suggests that the curvature of the streamlines of the flow over the crest of the trapezoidal profile weir has significant impact on the discharge capacity of this weir.

4.2 Model Results for Rough Bed

In practice most of the trapezoidal profile weirs have rough surfaces or they may develop rough flow boundaries over a period of time. It is important to assess the performance of the model for flow situation with rough boundaries. Assuming that the effects of roughness on the flow are purely frictional roughness, all parameters related to roughness can be substituted by the Nikuradse equivalent sand roughness height, k_S. According to Schlichting (1960), the Nikuradse equivalent sand roughness can be written as

$$k_S = k \exp(3.4 - 0.40C),$$

in which k is the roughness height and C is a constant for the logarithmic velocity profile equation and equal to $u(z)/U * -5.75 \log(z/k)$, $u(z)$ is flow velocity at z and $U *$ is the shear velocity. In this study, based on the observed velocity profiles upstream of the weir model $k_S = 1.864 \, \text{mm}$ was determined and this value was used to estimate the Manning roughness coefficient of the simulated rough bed. For flow over rough surfaces, it is very difficult to obtain a precise definition of the distance of the datum from the surface of the roughness for flow depth measurement. Approximate method based on velocity distribution can be used to estimate the location of this datum. Extrapolating the plot of $u(z)$ versus $\log(z)$ for the observed velocity distribution in the log region yields a datum level of 0.28 mm below the surface of the wire screen at $u(z) = 0$.
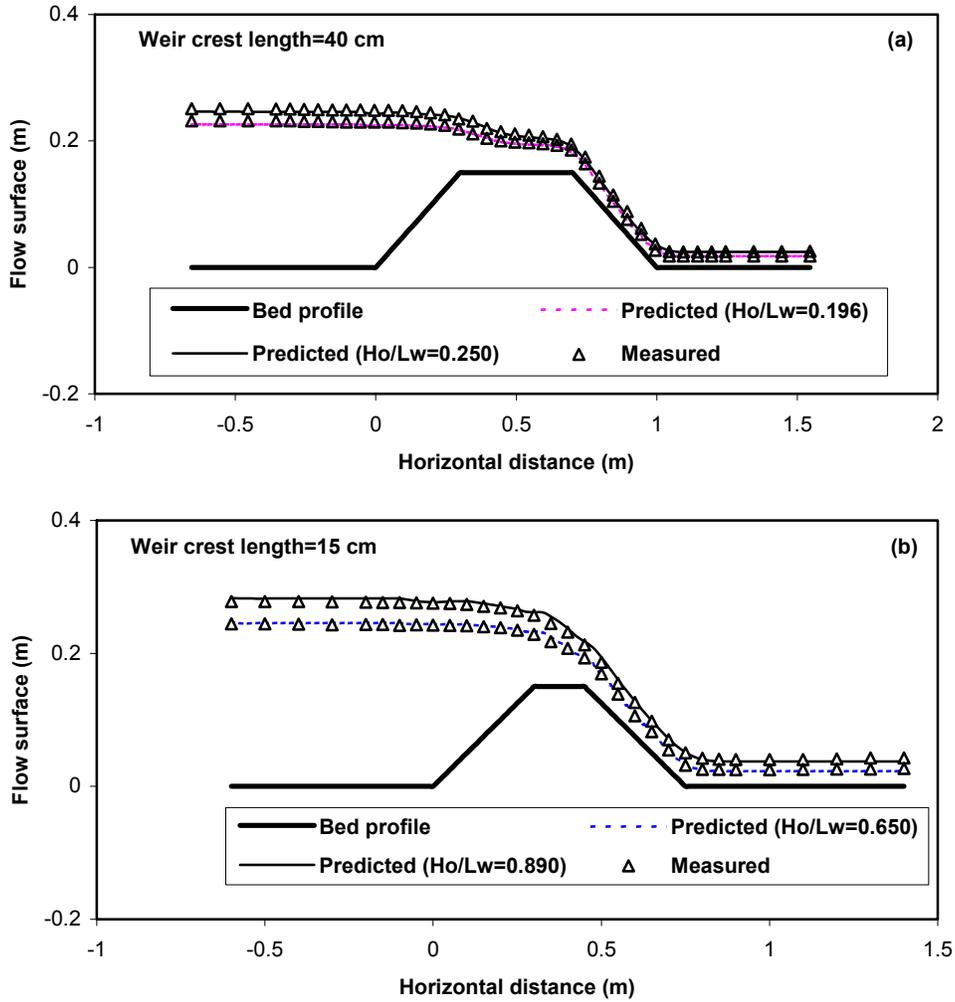
Figure 2 Flow surface profiles (smooth bed): (a) Broad- crested weir; (b) Short-crested weir.
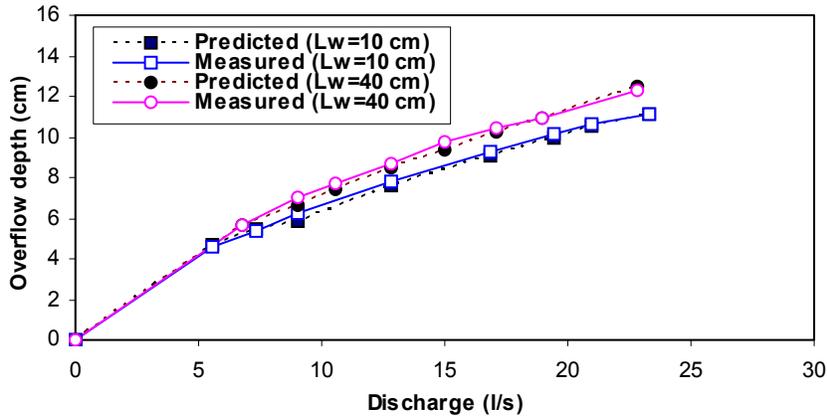
Figure 3 Comparison between discharge rating curves for short- and broad-crested weirs (smooth bed).

Figure 4 illustrates the transcritical flow profiles and a rating curve simulated by the model for a rough bed flow situation. The figure also compares the model results with experimental data. The

model reproduces the trend of the flow profiles and rating curve correctly and the results are in good agreement with experimental data. As in the previous case, the model predicts the flow surface profiles accurately for the entire flow region of the flow problems.
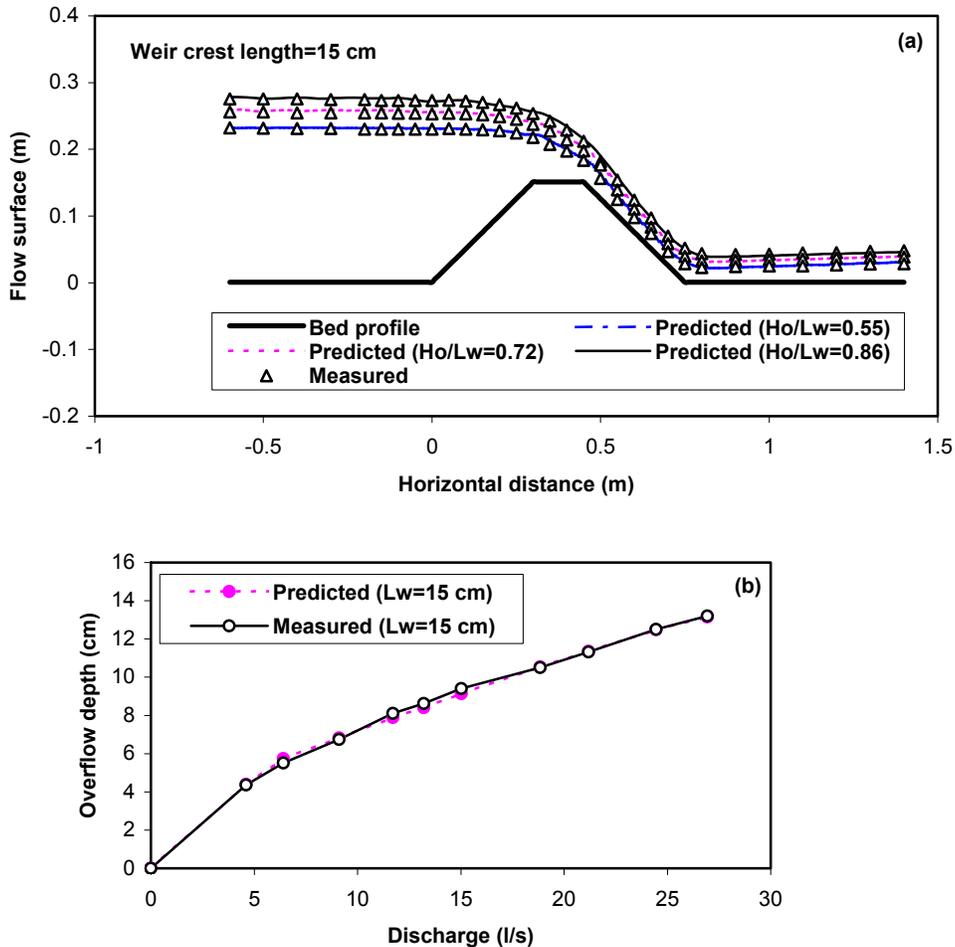
Figure 4 Comparison of predicted and measured results (rough bed): (a) Flow surface profiles; (b) Discharge rating curve.

The mean and standard deviation of the relative percentage errors of the predicted overflow depths was computed for the considered flow cases. The relative percentage error, at a given discharge, is defined as $(h_c - h_m)/h_m \times 100$, where h_c and h_m are the computed and measured overflow depths respectively. The mean and standard deviation analysis results showed that the model slightly underestimated the overflow depths for the short- and broad-crested weir flow situations. However, the magnitudes of the relative errors for all cases of the flow are very small. The overall mean and standard deviation of the errors for all flows are -0.518% and 2.49% respectively.

5.   CONCLUSIONS

A one-dimensional model was developed using a higher-order Boussinesq-type equation to simulate transcritical flow over short- and broad-crested trapezoidal profile weirs with smooth and rough

beds as well as to predict the upstream overflow depths for the given discharges. Finite difference approximations were used to discretise the flow equation. The Newton-Raphson iterative method with a numerical Jacobian matrix was applied for the solutions of the resulting nonlinear algebraic equations. Comparison of the numerical prediction results with experimental data was also presented. A good agreement was observed between the predicted and measured values for smooth and rough beds flow situations. The model predicts accurately the flow surface profiles irrespective of the curvature of the streamlines, demonstrating the capability of the model for handling flow situations with predominant non-hydrostatic pressure distribution effects. Also, the model results demonstrate the detailed dependence of the global flow characteristics of the trapezoidal profile weirs on the curvature of the streamlines. The existing flow model for developing discharge rating curves under free flow condition, which is valid only for broad-crested weir, was extended using the Boussinesq-type momentum equation model.

ACKNOWLEDGMENTS

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