Use of the programs FOURIER, CNOIDAL and STOKES for steady waves

John D. Fenton

Abstract

Latest change: In June 2019 I added an extra file to those output: SOLUTION-FLAT.RES, which is a results file which is rather more easily read automatically – see §7.2.

This document has accompanied the Fourier package of programs for several years. In February 2012 both programs and this document received a major upgrade, and in May 2013 another upgrade, plus incorporation of Stokes and cnoidal packages.

In a 2016 modification of this document, the introduction was re-written, putting the Fourier approach in context, and introducing the Stokes-Ursell number for the boundary between Stokes and cnoidal theories.

In September 2018 I realised that sometime in recent years I had modified the program to allow for the specification of infinite depth, but I had not noted this on the website or in this document.

Some years ago I upgraded both my cnoidal theory and Stokes theory packages, and distribute them with the Fourier package, to provide something of a check on both. The operation of the cnoidal package is described in §8 (with some program details in Appendix C), while the operation of the Stokes package is described in §9. They use the same data files as the Fourier program, and so implementation is simple. For two relatively low waves the results of computations are presented, and for the long wave case program CNOIDAL agrees very closely with FOURIER, while for the shorter wave case, so does the program STOKES. For a higher wave, described in the original experimental paper as being close to breaking, the two theories still agreed well. Although the packages based on theory can be used as a check, as done here, in general the Fourier program is to be preferred.

This document:	http://johndfenton.com/Steady-waves/Instructions.pdf
Home page of the programs:	http://johndfenton.com/Steady-waves/Fourier.html
The FOURIER package:	http://johndfenton.com/Steady-waves/Fourier.zip
The CNOIDAL package:	http://johndfenton.com/Steady-waves/Cnoidal.zip
The STOKES package:	http://johndfenton.com/Steady-waves/Stokes.zip

Both CNOIDAL and STOKES packages should be installed in a sub-directory of that where FOURIER is, so that they can read the same data files.

Recent revision history of FOURIER and this document **Instructions.pdf**	
20 December 2018	Georgii Bocharov of COWI noted that in comment lines in Flowfield.res, where a value of X/d should have been given, values of kX were printed. Files INOUT.CPP and the program FOURIER.EXE have been changed.
7 June 2019	I have added an extra file to those output: SOLUTION-FLAT.RES, which is a results file which is rather more easily read automatically – see §7.2
24 July 2019	Thomas Lykke Andersen noted that for very long waves, say 70 times the water depth, the cnoidal program had problems evaluating the elliptic functions in such an extreme limit. The cnoidal program has been modified as described in Appendix C.5, and the software simplified using accurate approximations for all elliptic functions and integrals.
31 October 2022	Ibrahim Konuk noted that in the C program a couple of variables defined were not necessary and that for deep water waves a quantity in the output in a Gnuplot comment line was not defined. All corrected. I also modified the Gnuplot files so that the graphical output mode "qt" is used, rather than "windows". For screen viewing it is better.

Table of Contents

1. Introduction

Throughout coastal and ocean engineering the convenient model of a steadily-progressing periodic wave train is used to give fluid velocities, surface elevations and pressures caused by waves, even in situations where the wave is being slowly modified by effects of viscosity, current, topography and wind or where the wave propagates past a structure with little effect on the wave itself. In these situations the waves do seem to show a surprising coherence of form, and they can be modelled by assuming that they are propagating steadily without change, giving rise to the so-called steady wave problem, which can be uniquely specified and solved in terms of three physical length scales only: water depth, wave length and wave height. In many practical problems it is not the wavelength which is known, but rather the wave period, and in this case, to solve the problem uniquely or to give accurate results for fluid velocities, it is necessary to know the current on which the waves are riding. In practice, the knowledge of the detailed flow structure under the wave is so important that it is usually considered necessary to solve accurately this otherwise idealised model.

The main theories and methods for the steady wave problem which have been used are: Stokes theory, an explicit theory based on an assumption that the waves are not very steep and which is best suited to waves in deeper water; and cnoidal theory, an explicit theory for waves in shallower water. The accuracy of both depends on the waves not being too high. In addition, both have a similar problem, that in the inappropriate limits of shallower water for Stokes theory and deeper water for cnoidal theory, the series become slowly convergent and ultimately do not converge.

An approach which overcomes this is the Fourier approximation method, which does not use series expansions based on a small parameter, but obtains the solution numerically. It could be described as a nonlinear spectral approach, where a series is assumed, each term of which satisfies the field equation, and then the coefficients are found by solving a system of nonlinear equations. This is the basis of the computer program FOURIER. It has been widely used to provide solutions in a number of practical and theoretical applications, providing solutions for fluid velocities and pressures for engineering design. The method provides accurate solutions for waves up to very close to the highest.

A review and comparison of the methods is given in Sobey, Goodwin, Thieke & Westberg (1987) and Fenton (1990).

The aim of this article is to

- present an introduction to the Fourier method ,
- describe the data format required by the program FOURIER
- describe the output files which are produced and how they might be used, including some graph-plotting files,
- to describe the basis of the Fourier method and the numerical techniques used,
- and to describe the use of the STOKES and CNOIDAL programs for comparison.

2. History

There have been two main analytical theories for solution of the problem of steadily-progressing water waves. The first method was developed by Stokes (1847). The basic form of the solution is to use a Fourier series which is capable of accurately approximating any periodic quantity, provided the coefficients in that series can be found. The analytical solution is obtained by using perturbation expansions for the coefficients in the series and solving linear equations at each order of approximation (Fenton 1985). The other theory is due to Korteweg and de Vries, who in 1895 used an expansion in shallowness developed by Boussinesq and Rayleigh, but obtained periodic solutions which they termed ”cnoidal” because the surface elevation is proportional to the square of the Jacobian elliptic function $\mathrm{cn}()$. The cnoidal solution shows the familiar long flat troughs and narrow crests of real waves in shallow water. In the limit of infinite wavelength, it describes a solitary wave. Since Korteweg and de Vries there have been a number of presentations of cnoidal theory, for higher orders consisting of series of cn functions. A review article of cnoidal theory was given by Fenton (1999).

For high waves, both Stokes and cnoidal series have problems with convergence. A reasonable procedure, then, is to calculate the coefficients by solving the full nonlinear equations numerically. This approach would be expected to be more accurate than either of the perturbation expansion approaches, Stokes and cnoidal theory, because its only approximations would be numerical ones, and not the essential analytical ones of the perturbation methods. Also, increasing the order of approximation would be a relatively trivial numerical matter without the need to perform extra mathematical operations. This approach originated with Chappelear (1961). He assumed a Fourier series in which each term identically satisfied the field equation throughout the fluid and the boundary condition on the bottom. The values of the Fourier coefficients and other variables for a particular wave were then found by numerical solution of the nonlinear equations obtained by substituting the Fourier series into the nonlinear boundary conditions. He used the velocity potential ϕ for the field variable and instead of using surface elevations directly he used a Fourier series for that too. Dean (1965) instead used the stream function ψ for the field variable and point values of the surface elevations, and obtained a rather simpler set of equations and called his method "stream function theory". Rienecker & Fenton (1981) presented a method based exclusively on Fourier approximation, whereas earlier work had used other lower-order numerical methods in part. The nonlinear equations were solved by Newton's method. The presentation emphasised the importance of knowing the current on which the waves travel if the wave period is specified as a parameter.

A simpler method and computer program was given by Fenton (1988), where the necessary matrix of partial derivatives was obtained numerically. In application of the method to waves which are high, in common with other versions of the Fourier approximation method (Dalrymple & Solana 1986), it was found that it is sometimes necessary to solve a sequence of lower waves, extrapolating forward in height steps until the desired height is reached. For very long waves all these methods can occasionally converge to the wrong solution, that of a wave one third of the length, which is obvious from the Fourier coefficients which result, as only every third is non-zero. This problem can be avoided by using a sequence of height steps.

It is possible to obtain nonlinear solutions for waves on shear flows for special cases of the vorticity distribution. For waves on a constant shear flow, Dalrymple (1974*a*), and a bi-linear shear distribution (Dalrymple 1974*b*) used a Fourier method based on the approach of Dean (1965). The ambiguity caused by the specification of wave period without current seems to have been ignored, however.

A review article of all theories was given by Fenton (1990). In this document we present the irrotational Fourier theory and describe the program that accompanies it.

3. The applicability of analytical and numerical theories

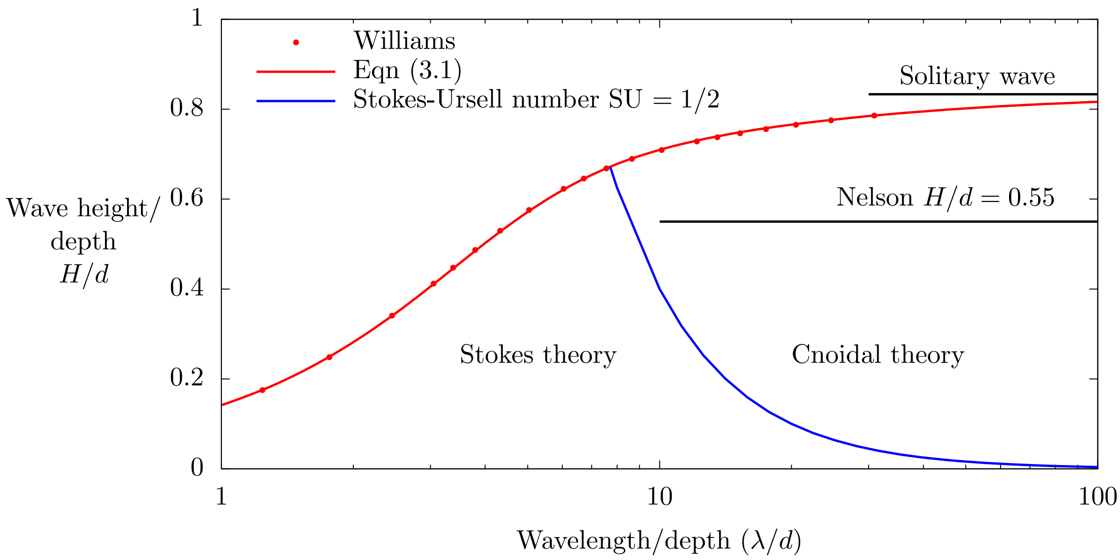

Figure 3-1. The region of possible steady waves, showing the theoretical highest waves (Williams) and the fitted equation (3.1), as well as other results described in the text.

The range over which periodic solutions for waves exist is given in Figure 3-1, which shows limits to the existence of waves determined by computational studies. The highest waves possible are shown by the red line, which is the approximation to the results of Williams (1981), presented as equation (32) in Fenton (1990):

$$\frac{H_m}{d}=\frac{0.141063\,\frac{\lambda}{d}+0.0095721\left(\frac{\lambda}{d}\right)^2+0.0077829\left(\frac{\lambda}{d}\right)^3}{1+0.0788340\,\frac{\lambda}{d}+0.0317567\left(\frac{\lambda}{d}\right)^2+0.0093407\left(\frac{\lambda}{d}\right)^3}, \tag{3.1}$$

where H_m is maximum crest-to-trough wave height, λ is length, and d is mean depth.

Slightly worryingly, Nelson (1987 and 1994) has shown from many experiments in laboratories and the field, that the maximum wave height achievable in practice is actually only $H_m/d = 0.55$. Further evidence for this conclusion is provided by the results of Le Méhauté, Divoky & Lin (1968), whose maximum wave height tested was $H/d = 0.548$, described as "just below breaking". It seems that there may be enough instabilities at work that real waves propagating over a flat bed cannot approach the theoretical limit given by equation (3.1).

Stokes theory: In Fenton (1985) it was shown that, whereas the nominal expansion parameter for Stokes' theory was $\varepsilon = kH/2$ ($k = 2\pi/\lambda$ the wave number), in the long wave limit the effective expansion parameter was actually $\varepsilon/\left(kd\right)^3$, thereby re-discovering something that Stokes knew, that for his theory to be accurate $\varepsilon/\left(kd\right)^3$ should be small. The relatively low order of the approximation and the nature of the effective expansion parameter mean that Stokes theory is accurate for waves that are not long and not very high.

Cnoidal theory: A higher-order cnoidal theory was obtained by Fenton (1979). Whereas the nominal expansion parameter was H/h, where h is the trough depth of the water, it was shown that the effective expansion parameter is actually $\left(H/h\right)/m$, where m is the elliptic parameter in the theory. For solitary waves $m = 1$. For shorter waves, m becomes smaller, hence, in this way the cnoidal theory breaks down in deep water (short waves) in a manner complementary to that in which Stokes theory breaks down in shallow water (long waves). The 1979 paper presented results for fluid velocity which fluctuated wildly and were not accurate for high waves. In Fenton (1990) the series were expressed in terms of a shallowness parameter and much better results were obtained for fluid velocities.

Boundary of applicability between Stokes and cnoidal theory: Isobe, Nishimura & Horikawa (1982) presented a unified view of Stokes and cnoidal theories. They proposed a boundary between areas of application of Stokes and cnoidal theory of Ur = 25, where Ur is the Ursell number,

$$\mathrm{Ur}=\frac{H\lambda^2}{d^3}=\frac{H/d}{(d/\lambda)^2}=\frac{\text{Measure of Nonlinearity}}{\text{Measure of Shallowness}}. \tag{3.2}$$

Fenton (1990) proposed a boundary, based on a number of computations, however Hedges (1995) showed that a more satisfactory and simpler boundary was $\mathrm{Ur} = 40$. The only problem with that is an aesthetic one – that it is not a nice number below which something is deemed to be small. Here we produce an alternative approach. In his original work, Stokes' required for his theory to be valid, that $\frac{1}{2}kH/(kd)^2$ be small. It is easily shown that that Stokes parameter is simply $\mathrm{Ur}/8\pi^2$, so Stokes preceded Ursell by some 100 years. There is a strong temptation to rename the number after Stokes; unfortunately there already is a Stokes number in viscous flow. The problem can be solved in a sense if we call this the Stokes-Ursell number SU:

$$\mathrm{SU}=\frac{kH/2}{\left(kd\right)^3}=\frac{1}{2}\frac{H/k^2}{d^3}=\frac{1}{8\pi^2}\frac{H\lambda^2}{d^3}\approx\frac{\mathrm{Ur}}{80}. \tag{3.3}$$

Now, with Hedges' proposed boundary $\mathrm{Ur} = 40$, we have the pleasant result that Stokes' theory should not be applied beyond a value of $\mathrm{SU} = \frac{1}{2}$ – a reasonable value for the limit beyond which a quantity ceases to be small.

All three programs, FOURIER, STOKES and CNOIDAL output the value of SU, in case it is of interest.

Fourier approximation: Results from Fourier methods show that accurate solutions can be obtained with Fourier series even for waves close to the highest given by equation (3.1), and they seem to be the best way of solving any steady water wave problem where accuracy is important. Sobey et al. (1987) made a comparison between different versions of the numerical methods. They concluded that there was little to choose between them.

Generally the Fourier approach works well up to about 98% of the maximum height for a given wavelength/depth. In application of the method to waves which are high and long, in common with other versions of the Fourier

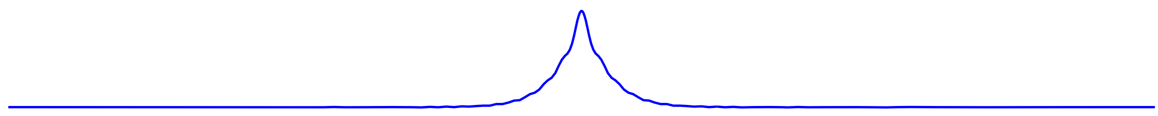

Figure 3-2. Free surface for a wave of length $L/d = 50$ and a height of $H/d = 0.786$, 98% of the maximum height possible for that length. There were $N = 70$ terms in the Fourier series, and the highest wave was computed from a sequence of 20 waves, using initial solutions extrapolated from two previous solutions.

approximation method, the Fourier method may converge to a wave of 1/3 of the wavelength (Dalrymple & Solana 1986, with comments by Fenton and Sobey noted in the References), but this can be remedied by solving for lower waves of the same length and stepping upwards in height (Fenton 1988), as is done in the present program. The Fourier approach does break down in the limit of very long and very high waves, when the spectrum of coefficients becomes broad-banded and many terms have to be taken, as the Fourier approximation has to approximate both the short sharp crest region and the long trough where very little changes. However, over a very wide range of lengths and heights, the Fourier method works well. For example Figure 3-2 shows results for the surface profile using the present Fourier program for a wave of length $L/d = 50$ and a height of $H/d = 0.786$, 98% of the maximum height possible for that length. It can be seen with the very long wave and the crest approaching sharp the Fourier approximation has to work very hard indeed, and some slight oscillations are visible, but it has obtained a solution, and there were no such oscillations in the fluid velocities.

Fenton (1995) developed a numerical cnoidal theory so that very long waves could be treated without difficulty, however for wavelengths as long as 50 times the depth, the Fourier method provides good solutions.

4. The physical problem

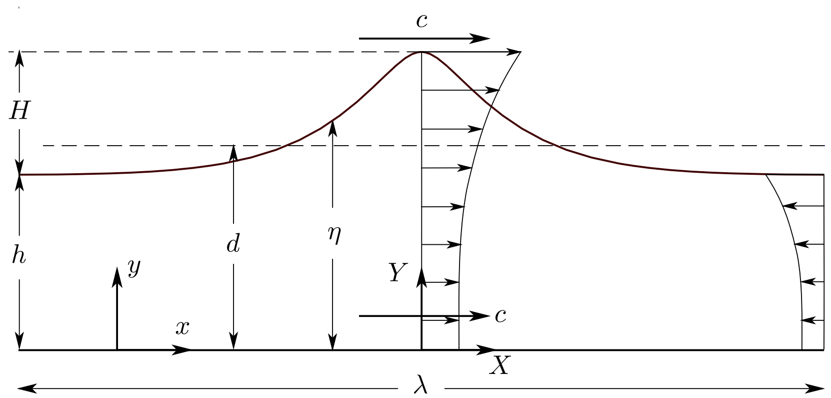

Figure 4-1. One wave of a steady train, showing principal dimensions, co-ordinates and velocities

The problem considered is that of two-dimensional periodic waves propagating without change of form over a layer of fluid on a horizontal bed, as shown in Figure 4-1. A co-ordinate system (x, y) has its origin on the bed, and waves pass through this frame with a velocity c in the positive x direction. It is this stationary frame which is the usual one of interest for engineering and geophysical applications. Consider also a frame of reference (X, Y) moving with the waves at velocity c, such that $x = X + ct$, where t is time, and $y = Y$. It is easier to solve the problem in this moving frame in which all motion is steady and then to compute the unsteady velocities. If the fluid velocity in the (x, y) frame is (u, v), and that in the (X, Y) frame is (U, V), the velocities are related by $u = U + c$ and $v = V$.

The physical dimensions are: crest-to-trough wave height H, mean water depth d, wavelength λ, and trough depth h. The surface elevation η is relative to the sea bed, so that it is the total depth of water at any place and time.

5. The program FOURIER.EXE

All the files necessary can be found at http://johndfenton.com/Steady-waves/Fourier.zip. The executable program is FOURIER.EXE.

6. Input data

It is well-known that a steadily-progressing periodic wave train is uniquely specified by three length scales, the water depth d, the wave height H, and the wavelength λ, or, in terms of only two dimensionless quantities involving these, such as dimensionless wave height H/d and dimensionless wavelength λ/d. The program allows for the specification of these, however in many practical situations it is not the wavelength which is known, but the wave period τ. If this is the case, it is not enough to uniquely specify the wave problem, as if there is a current, any current, then the period will be Doppler-shifted. Hence, it is necessary also to specify the current in such cases. The value of this current will also affect the horizontal velocity components, and users of the program should be aware of this and if it is unknown, some maximum and minimum values might be tried and their effects evaluated.

Usually all input data are to be specified in terms non-dimensionalised with respect to gravitational acceleration g and mean depth d, however an option is for water of infinite depth, to specify a value of H/λ.

There are three files necessary, which should be in the same directory as FOURIER.EXE:

6.1 DATA.DAT

The wave data is of the form as given in column 1 of Table 6-1. Any other information, such as that in column 2, can be placed after column 1 on each line, such as we have done here, to label each line. A blank is also allowed. The same data file is read by FOURIER, CNOIDAL, and STOKES. The latter two programs interpret the number N as the order of the theory to use, described in §6.1.5 below.

Test wave (A title line to identify each wave)	
0.5	H/d: if a negative value is given here it is interpreted as $-H/\lambda$ and the depth is infinite.
Wavelength	Measure of length: "Wavelength" or "Period"
10.	Value of that length: λ/d or $\tau\sqrt{g/d}$ respectively
1	Current criterion, 1 for Euler, or 2 for Stokes
0.	Current magnitude, $\bar{u}_1/\sqrt{gd}$ or $\bar{u}_2/\sqrt{gd}$
20	N: Number of Fourier components or order of Stokes or cnoidal theory
1	Number of height steps to reach H/d
FINISH	Must be used to tell the program to stop - the file can continue after this

Table 6-1. Form of data to be supplied for each wave

Here we describe the nature of each element of the input data.

6.1.1 Description

A line containing any identifier or description of the wave, up to 100 characters

6.1.2 Wave height

The relative wave height H/d is specified. There is a formula for the maximum wave height H_m/d for a particular wavelength λ/d, given as equation (3.1). In many problems, where it is the period that is specified it is not possible to calculate the highest possible wave height *a priori*. The program, after it solves a wave, prints out the theoretical maximum H_m/d for the calculated wave length. The user could then reconsider the value of H/d to specify. If a negative value is given here it is interpreted as $-H/\lambda$ and the depth is infinite.

6.1.3 Wavelength or Period

If "Wavelength" is chosen, then a value of λ/d is then specified in the next line; if "Period" then a value of dimensionless period $\tau\sqrt{g/d}$ is to be given.

That requires a value of gravitational acceleration g, which is a function of latitude – because of the apparent centrifugal acceleration due to the rotation of the earth, g is smaller nearer the equator than at the poles. Figure 6-1 shows the variation. In many hydraulic applications $g = 10\ \mathrm{m\,s^{-2}}$ is quite acceptable, as 2% accuracy is enough, because the theories or measurements are only approximate. If three-figure accuracy were required in a calculation then

$$g = 9.806 - 0.026\,\cos(2\theta) \quad \text{in} \quad \mathrm{m\,s^{-2}},$$

where θ is latitude (in radians). A sensible step would be to use $g = 9.8\ \mathrm{m\,s^{-2}}$,which is accurate to 0.1%. The widespread use of $g = 9.81\ \mathrm{m\,s^{-2}}$, is a strange historical accident, apparently following European and North American textbooks, and is not justified. It can be seen in the figure that this is most appropriate to northern Europe, Mongolia, parts of Siberia, Canada, Las Islas Malvinas and Tierra del Fuego– and is not justified for most of the world's population!

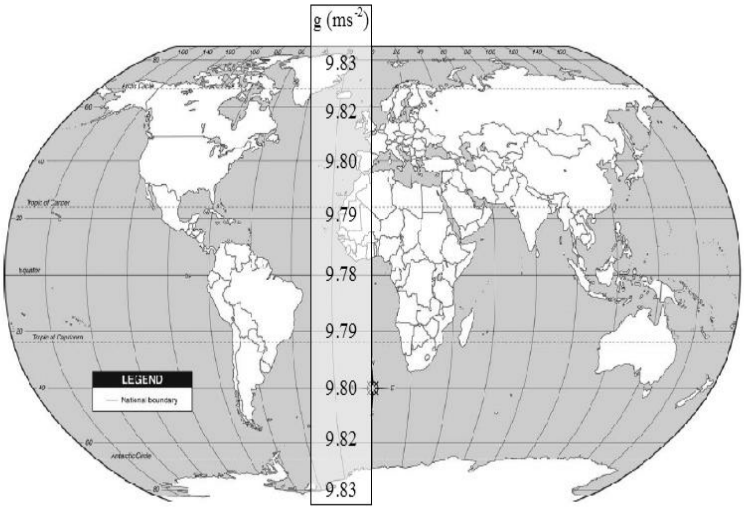

Figure 6-1. Variation of gravitational acceleration g ($\mathrm{m\,s^{-2}}$) with latitude

6.1.4 Current

This is described in more detail in Appendix A.2 below. There are actually two definitions of currents, the first, identified by 1 here is the "Eulerian mean current", the time-mean horizontal fluid velocity at any point denoted by $\bar{u}_1$, the mean current which a stationary meter would measure. In irrotational flow this is constant everywhere. A second type of mean current is the depth-integrated mean current, the "mass-transport velocity", which we denote by $\bar{u}_2$. If there is no mass transport, such as in a closed wave tank, $\bar{u}_2 = 0$. Usually the overall physical problem will impose a certain value of current on the wave field, thus determining the wave speed. To apply the methods of this theory if wave period rather than length is known, to obtain a unique solution it is necessary to specify both the nature (1 or 2) and magnitude of that current. If the current is unknown, any horizontal velocity components calculated are approximate only.

6.1.5 Number of Fourier components or Order of theory

FOURIER: The number of terms in the series N is the primary computational parameter in the program. The program now has no limit (previously it was $N = 32$), but for many problems, $N = 10$ is enough – results show that accurate solutions can be obtained with Fourier series of 10-20 terms, even for waves close to the highest, although for longer and higher waves it may be necessary to increase N. The adequacy of the particular value of N used can be monitored by examining the output file SOLUTION.RES, where the spectra of Fourier coefficients obtained as part of the solution is presented, the B_j which are at the core of the method, as presented in equation (A-5) for $j = 1, \ldots, N$, and the Fourier coefficients of the computed free surface, the E_j as presented in equation (B-9). The B_j decay rather more rapidly than do the E_j. The value of E_N should be sufficiently small (less than 10^{-4} say) that there would be no identifiable high-frequency wave apparent on the surface plotted from the solution (*cf.* Figure 3-2 above).

STOKES: If N is in the range 1 to 5, that is the order of the theory used. Or, if the data file has a value of N greater than 5, such as would be used for the Fourier program, the Stokes program resets it automatically to $N = 5$, the most accurate version of the Stokes theory contained in this package.

CNOIDAL: If N is in the range 1 to 5, that is the order of the theory used. If one uses $N > 5$, such as for the Fourier program, the Cnoidal program resets it automatically to $N = 6$, *and uses Aitken convergence enhancement for global quantities*, giving rather better results. This is recommended.

6.1.6 Number of height steps

In application of the method to waves which are high and long, in common with other versions of the Fourier approximation method, the Fourier method may converge to a wave of 1/3 of the wavelength (Dalrymple & Solana 1986, with comments by Fenton and Sobey noted in the References), but this can be remedied by solving for lower waves of the same length and stepping upwards in height (Fenton 1988). This occurrence of this phenomenon is made obvious from the Fourier coefficients which result, as only every third is non-zero. The present program overcomes this by solving a sequence of lower waves, extrapolating forward in height steps until the desired height is reached. For waves up to about half the highest $H \approx H_m/2$ it is not necessary to do this, and a value of 1 in the eighth line of the data file is all right, but thereafter it is better to take 2 or more height steps. For waves very close to H_m for a given length it might be necessary to take as many as 20. The evidence as to whether enough have been taken is provided by the spectrum, as noted above.

6.2 CONVERGENCE.DAT

This is a three-line file which controls convergence of the iteration procedure, for example:

Control file to control convergence and output of results
20 Maximum number of iterations for each height step; 10 OK for ordinary waves, 40 for highest
1.e-4 Criterion for convergence, typically 1.e-4, or 1.e-5 for highest waves

6.3 POINTS.DAT

This controls how much information is to be printed out afterwards to show the velocity and acceleration fields. For example:

Control output (for graph plotting *etc.*)
50 M, Number of points on free surface (the program clusters them near the crest)
8 Number of velocity/acceration profiles over half a wavelength to print out, including $x = 0$ and $x = \lambda/2$.
20 Number of vertical points in each profile, including points at bottom and surface.

7. Output files

The program produces output to the screen showing how the process of convergence is working. Three files are produced:

7.1 SOLUTION.RES

After a heading block, including the theoretical highest wave for this length of wave, and the Stokes-Ursell number, the program prints out the global parameters of the wave train, where all quantities are shown first non-dimensionalised with respect to ρ, g and k, where $k = 2\pi/\lambda$ is the wavenumber of the wave, and then the value non-dimensionalised with respect to g and depth d. The results are as listed in Table 7-1. Following the global parameters shown, the spectra of the velocity potential coefficients B_j and the surface elevation coefficients E_j are given, for $j = 1, \ldots, N$, the two corresponding coefficients on each row. These spectra should be checked, as suggested above, to ensure that the coefficients have become small enough that the solution has converged satisfactorily, and that it has not converged to one which is $1/3$ of the wavelength.

Quantity	Dimensionless w.r.t.		Reference	
	k	d	This document	Fenton (1988)
Water depth	kd	$d/d=1$		
Wave length	$k\lambda=2\pi$	λ/d		
Wave height	kH	H/d		
Wave period	$\tau\sqrt{gk}$	$\tau\sqrt{g/d}$		
Wave speed	$c\sqrt{k/g}$	$c/\sqrt{gd}$		
Eulerian current	$\bar{u}_1\sqrt{k/g}$	$\bar{u}_1/\sqrt{gd}$	(A-13)	Symbol c_E, p358
Stokes current	$\bar{u}_2\sqrt{k/g}$	$\bar{u}_2/\sqrt{gd}$	(A-14)	Symbol c_S, p359
Mean fluid speed	$\bar{U}\sqrt{k/g}$	$\bar{U}/\sqrt{gd}$	(A-5)	
Wave volume flux, $q=\bar{U}d-Q$	$q\sqrt{k^3/g}$	$q/\sqrt{gd^3}$		p359
Bernoulli constant, $r=R-gd$	rk/g	r/gd		p360
Volume flux	$Q\sqrt{k^3/g}$	$Q/\sqrt{gd^3}$	(A-3)	p360
Bernoulli constant	Rk/g	R/gd	(A-4)	p360
Momentum flux	$Sk^2/\rho g$	$S/\rho gd^2$		p362
Impulse	$I\sqrt{k^3}/\rho\sqrt{g}$	$I/\rho\sqrt{gd^3}$		p362
Kinetic energy	$Tk^2/\rho g$	$T/\rho gd^2$		p362
Potential energy	$Vk^2/\rho g$	$V/\rho gd^2$		p362
Mean square of bed velocity	u_b^2k/g	u_b^2/gd		p362
Radiation stress	$S_{xx}k^2/\rho g$	$S_{xx}/\rho gd^2$		p362
Wave power	$Fk^{5/2}/\rho g^{3/2}$	$F/\rho g^{3/2}d^{5/2}$		p362
Fourier coefficients (dimensionless)	B_j	E_j	(A-5,B-5)	p360, p362
	B_1	E_1		
	$\ldots$	$\ldots$		
	B_N	E_N		

Table 7-1. Quantities printed out at the head of file SOLUTION.RES

7.2 SOLUTION-FLAT.RES

In June 2019 I added a results file, containing the same results as SOLUTION.RES, and itemised in Table 7-1, but with a flat structure so that the results are much more easily automatically read. The structure is:

- Single text line containing the name of the wave as specified in DATA.DAT
- Single text line stating that the following lines contain the solution
- 19 rows, each with an identifying integer plus two e-format floating point numbers giving the solution (1) non-dimensionalised by g & k, and (2) g & d, plus a 40-character description of the physical variable.
- N, the numerical value of the number of terms in the Fourier series, plus some explanatory text ,
- N rows, each containing the number of the row j and two e-format floating point numbers giving the dimensionless Fourier coefficients B_j in the series for ψ (equation A-5) and E_j in the series for η (equation B-9)

7.3 SURFACE.RES

This file contains co-ordinates of points on the surface, all given non-dimensionally with respect to the water depth d. It contains three columns, the first giving the X co-ordinates over a range from trough-crest-trough, clustered quadratically near the crest for plotting purposes, $X_i/d=(i/(M/2))^2\,\lambda/d/2$ for $i=-M/2..+M/2$, where M is the number of surface points defined in the description of file POINTS.DAT above. The second column is the free surface elevation (total water depth) η_i/d. The third column contains a check on calculations, the computed value of the pressure on the surface $p_i/\rho gd$ from equation (B-10), which should be zero, being typically of the order of the last surface Fourier component E_N.

7.4 FLOWFIELD.RES

This contains a number of profiles of velocity components and time derivatives, the number of profiles and the number of points in each profile determined by file POINTS.DAT. For a sequence of (here equi-spaced) X/d values between 0 (crest) and $\lambda/d/2$ (trough), and then for each for a number of y/d from 0 (the bed) to the local free surface elevation η/d, quantities output on each line are shown in Table 7-2. Note that all are dimensionless with respect to g and d, the mean depth.

$\frac{y}{d}$	$\frac{u}{\sqrt{gd}}$	$\frac{v}{\sqrt{gd}}$	$\frac{\partial\phi/\partial t}{gd}$	$\frac{\partial u/\partial t}{g}$	$\frac{\partial v/\partial t}{g}$	$\frac{\partial u}{\partial x}\sqrt{\frac{d}{g}} = -\frac{\partial v}{\partial y}\sqrt{\frac{d}{g}}$	$\frac{\partial u}{\partial y}\sqrt{\frac{d}{g}} = \frac{\partial v}{\partial x}\sqrt{\frac{d}{g}}$	Bernoulli check, equation B-12

Table 7-2. Line of output in file FLOWFIELD.RES

7.5 Graphical output

The package includes a file that enables the plotting of data from the results files. When run with the GNUPLOT program (http://www.gnuplot.info/), the file FIGURES.PLT (which uses file SETOUTPUT.PLT) produces three figures as shown here in Figure 7-1.

8. The program CNOIDAL.EXE

8.1 Use

This program accompanies the program FOURIER, and it can be used to give a check of the results of that program for waves that are not too high. The program CNOIDAL.EXE should be placed in a subdirectory (maybe "Cnoidal"?) of the directory where FOURIER.EXE is placed. CNOIDAL.EXE reads the same data files DATA.DAT, for the wave data, and POINTS.DAT for control of the output. It looks for those files in the directory above where it is placed. It produces three files in its own directory, which are very similar to the files created by FOURIER. They are SOLUTION.RES, FLOWFIELD.RES, and SURFACE.RES. There is a Gnuplot file FIGURES-CNOIDAL-FOURIER.PLT also in that directory, which can be used for plotting. Also placed in the Cnoidal directory are all the C++ files necessary.

Note should be made here, that in line 7 of the data file DATA.DAT, what FOURIER reads as N, the number of terms in the Fourier series, is treated differently by CNOIDAL. In the range of 1 to 5, it gives the order of the theory to apply. If 6 or greater it defaults to 6 and uses the 5th order theory with Aitken enhancement of the series (see below, and which gives the most accurate results). Velocity components, however, just use 5th order theory, which was found to be best.

8.2 Results

Two waves were tested and results described here. The first is a low wave, so as to test both FOURIER.EXE and CNOIDAL.EXE together. The wave rides on a large current, so as to test the ability of both programs to handle that case. The wave characteristics are: height $H/d = 0.3$, period $\tau\sqrt{g/d} = 20$ (a long wave), and it rides on an Eulerian current $\bar{u}_1/\sqrt{gd} = 0.1$, or about 1/10 of the wave speed. All results from the two programs agreed to three decimal places, and are shown graphically in Figure 8-1. As the two theories use completely different means of approximation and programs, while using the same equations, the results are an encouraging demonstration that both programs are working well.

The second wave tested was a high and long wave, from Figure 9 of Le Méhauté et al. (1968), the highest and longest of all the waves they tested, with $H/d = 0.548$ and $\tau\sqrt{g/d} = 27.24$. The experiments were conducted in a closed wave tank, such that there is no net mass transport, giving the condition $\bar{u}_2 = 0$. They noted that that wave was "just below breaking", which provides further evidence for Nelson's assertion (Nelson, 1987 and 1994) that the maximum wave height achievable in practice is actually only $H_m/d \approx 0.55$. Figure 8-2 shows the results from FOURIER and CNOIDAL. Both describe the free surface very closely. However, there are some disagreements between the two for horizontal fluid velocity (the results from FOURIER are expected to be more accurate), and it can be seen that the agreement with experiment is not all that close. In the experiments the fluid velocities under

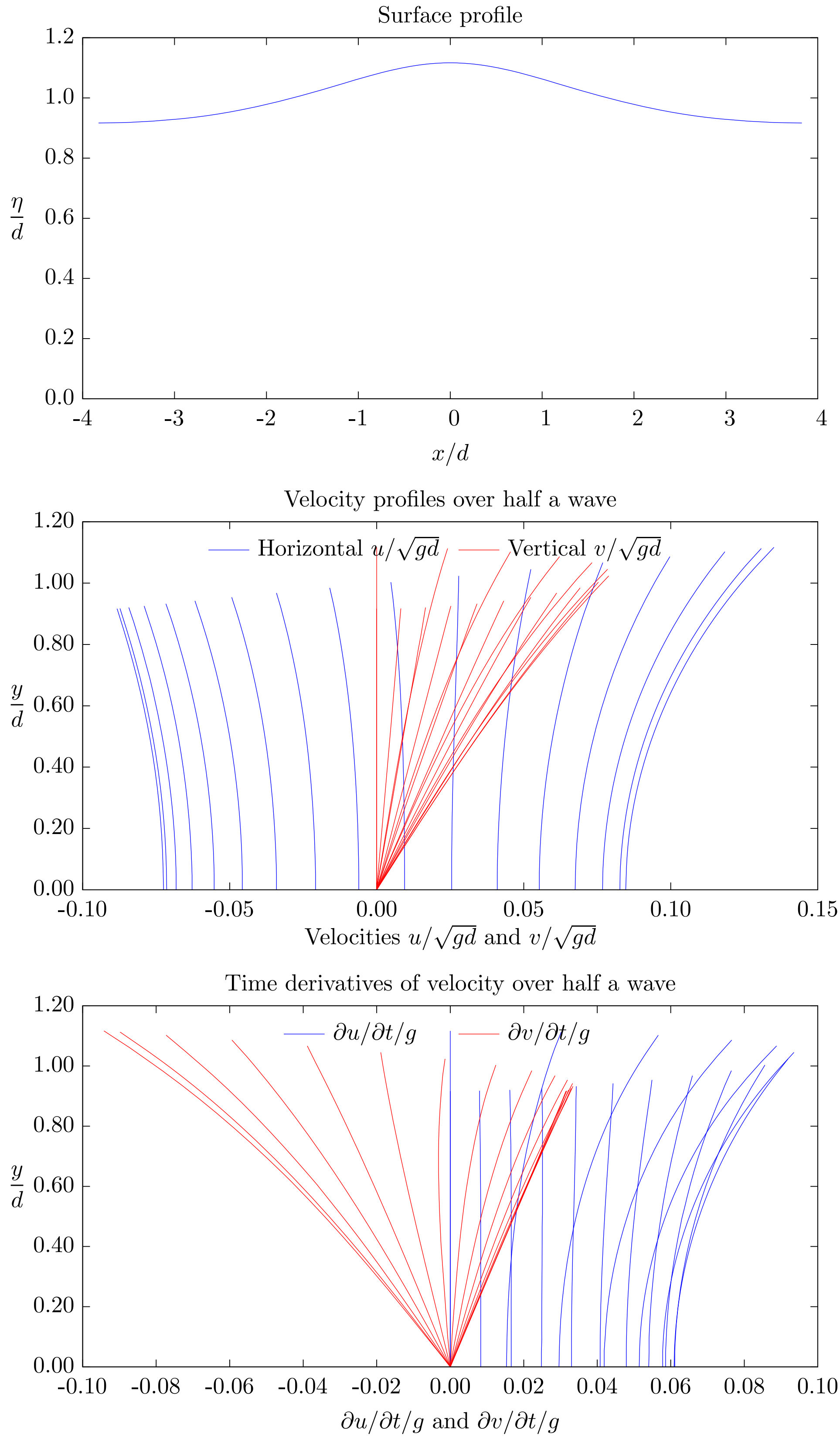

Figure 7-1. Figures obtained by Gnuplot from output files produced by FOURIER for a wave of height $H/d = 0.5$ and length $\lambda/d = 10$

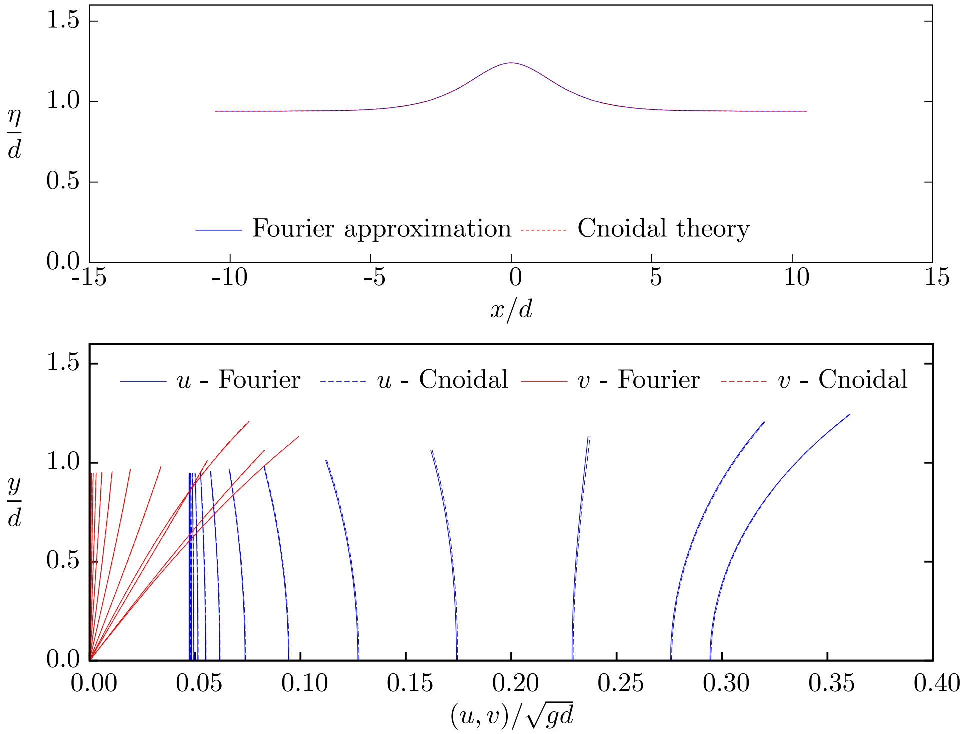

Figure 8-1. Results from the Fourier and Cnoidal programs for a low and long wave, $H/d = 0.3$, $\tau\sqrt{g/d} = 20$, Eulerian current $\bar{u}_1/\sqrt{gd} = 0.1$.

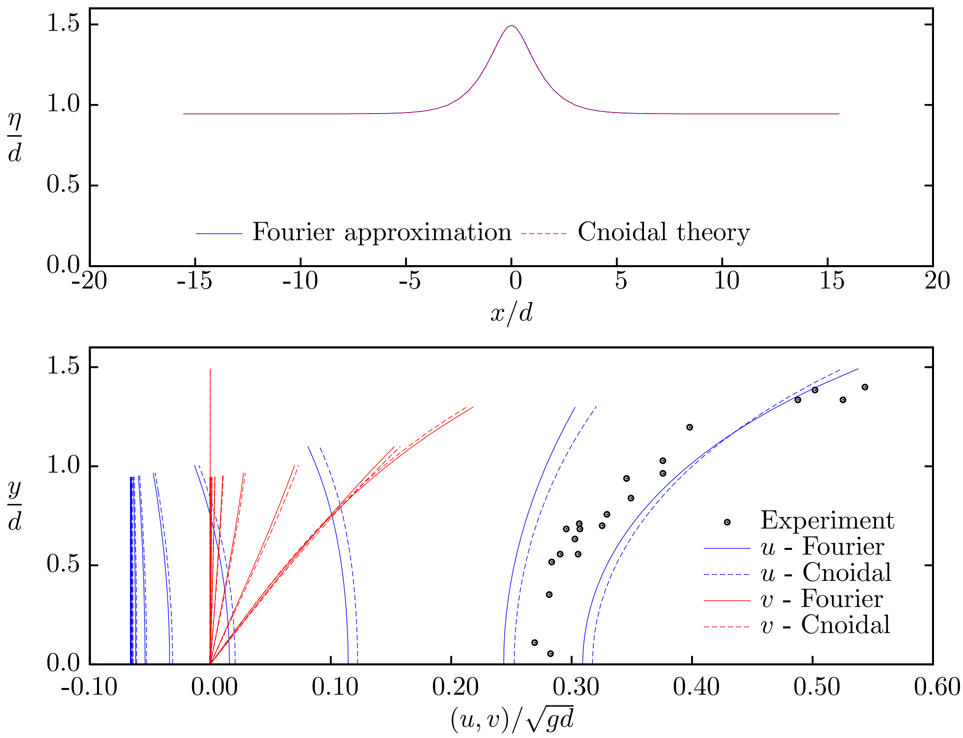

Figure 8-2. Results from the Fourier and Cnoidal programs for a high and long wave in a closed tank such that $\bar{u}_2 = 0$, with experimental horizontal crest velocities from Figure 9 of Le Méhauté et al. (1968)

the wave crests were measured by tracking of marker particle motions: if anything this would under-estimate the actual instantaneous velocities in the laboratory. Also, the experiments were not large, for this case a mean depth of $0.17\,\mathrm{m}$. Overall, given engineering uncertainties of real problems, the cnoidal program works well and could be used in practice if necessary, but probably more as a check of the Fourier program, and for waves not higher than about $H/d \approx 0.6$.

Theoretical and computational aspects of the cnoidal program are given in Appendix C.

9. The program STOKES.EXE

9.1 Use

This program accompanies the program FOURIER, and it can be used to give a check of the results of that program for waves that are not too high. The program STOKES.EXE should be placed in a subdirectory (maybe "Stokes"?) of the directory where FOURIER.EXE is placed. STOKES.EXE reads the same data files DATA.DAT, for the wave data, and POINTS.DAT for control of the output. It looks for those files in the directory above where it is placed. It produces three files in its own directory, which are very similar to the files created by FOURIER. They are SOLUTION.RES, FLOWFIELD.RES, and SURFACE.RES. There is a Gnuplot file FIGURES.PLT also in that directory, which can be used for plotting. Also placed in the Stokes directory are all the C++ files necessary.

Note should be made here, that line 7 of the data file DATA.DAT gives the order of the Stokes theory that will be used. A value greater than 5 will be automatically reset to 5. That 5th-order Stokes theory is as presented in Fenton (1985), and unlike the cnoidal theory, nothing will be presented here. The program output is exactly the same format as that of FOURIER.EXE, but appears in the directory where STOKES.EXE is located. It is just as if $N = 5$ had been used with FOURIER.EXE.

Whereas CNOIDAL.EXE makes use of Aitken transforms to improve convergence, STOKES.EXE does not do this. The program automatically applies 5th-order theory.

9.2 Results

A test wave was solved with both STOKES.EXE and FOURIER.EXE. It is the example wave used in Fenton (1985, p222), where wavelength is specified, $\lambda/d = 8.3333$, which is approaching the generally recognised limit of Stokes theory. With a relatively low wave height of $H/d = 0.3$, chosen so that the Stokes program should agree with the generally-more accurate Fourier method as a check that both are operating correctly, the program calculated a Stokes-Ursell number of $SU = 0.264$, indeed approaching the actual limit of applicability $SU \leqslant \frac{1}{2}$ described above. An arbitrary small current of $\bar{u}_1/\sqrt{gd} = 0.01$ was used (which affected only the horizontal velocities). Results are shown in Figure 9-1.

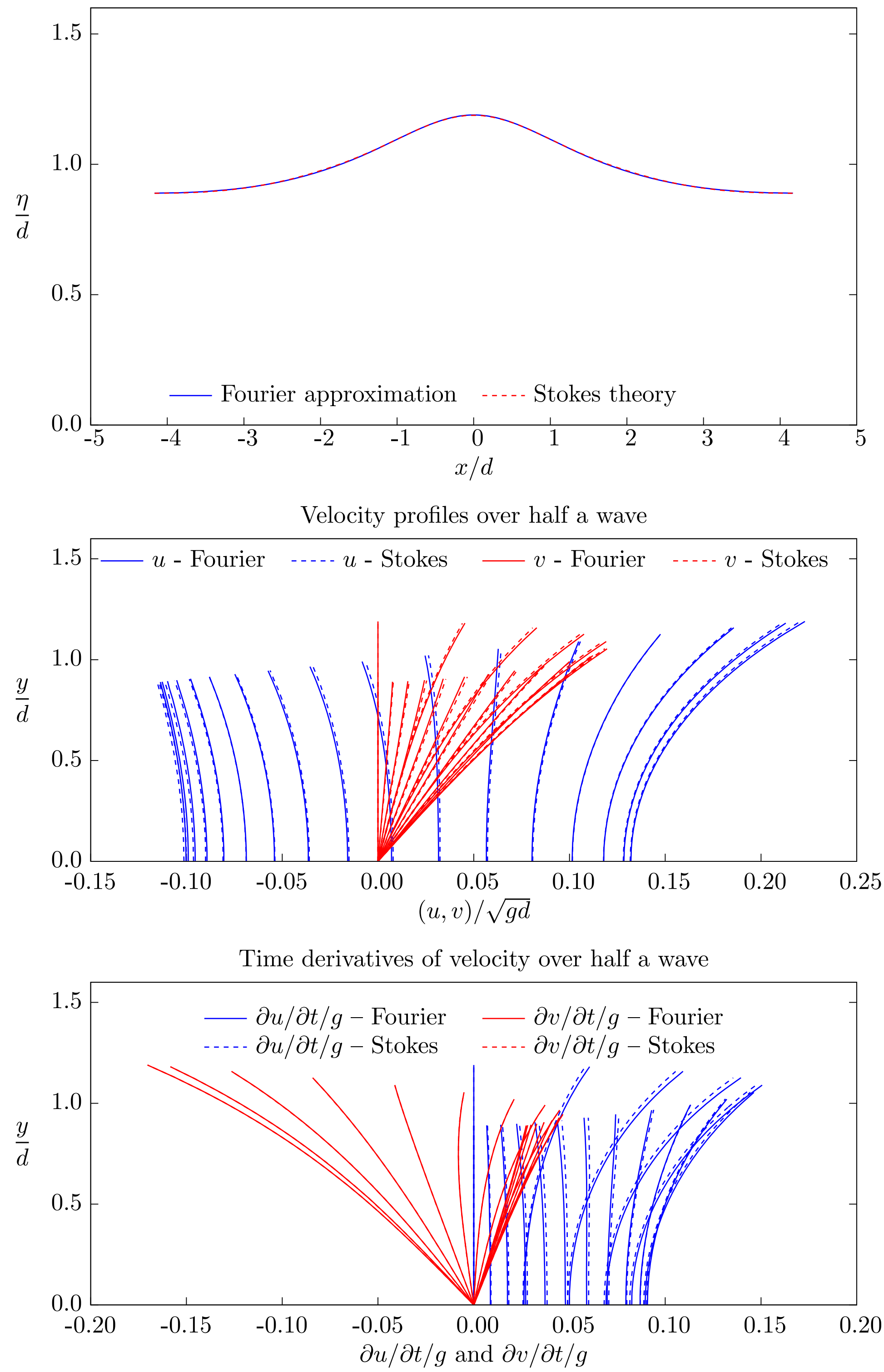

Figure 9-1. Figures obtained by Gnuplot from output files produced by FOURIER and STOKES for a wave of height $H/d = 0.3$ and length $\lambda/d = 8.3333$

References

Abramowitz, M. & Stegun, I. A. (1965), *Handbook of Mathematical Functions*, Dover, New York.

Chappelear, J. E. (1961), Direct numerical calculation of wave properties, *J. Geophys. Res.* **66**, 501–508.

Chappelear, J. E. (1962), Shallow-water waves, *J. Geophys. Res.* **67**, 4693–4704.

Dalrymple, R. A. (1974*a*), A finite amplitude wave on a linear shear current, *J. Geophys. Res.* **79**, 4498–4504.

Dalrymple, R. A. (1974*b*), Water waves on a bilinear shear current, in *Proc. 14th Int. Conf. Coastal Engng*, Copenhagen, pp. 626–641.

Dalrymple, R. A. & Solana, P. (1986), Nonuniqueness in stream function wave theory, *J. Waterway Port Coastal and Ocean Engng* **112**, 333–337. See Discussion in Vol 114 (1988) of the same Journal by J.D. Fenton, pages 110-112, and R.J. Sobey, pages 112-114.

Dean, R. G. (1965), Stream function representation of nonlinear ocean waves, *J. Geophys. Res.* **70**, 4561–4572.

Fenton, J. D. (1972), A ninth-order solution for the solitary wave, *J. Fluid Mechanics* **53**, 257–271.

Fenton, J. D. (1979), A high-order cnoidal wave theory, *J. Fluid Mechanics* **94**, 129–161.

Fenton, J. D. (1985), A fifth-order Stokes theory for steady waves, *J. Waterway Port Coastal and Ocean Engng* **111**, 216–234, Errata: Vol 113, p438. http://johndfenton.com/Papers/Fenton85d-A-fifth-order-Stokes-theory-for-steady-waves.pdf.

Fenton, J. D. (1988), The numerical solution of steady water wave problems, *Computers and Geosciences* **14**, 357–368.

Fenton, J. D. (1990), Nonlinear wave theories, in B. Le Méhauté & D. M. Hanes, eds, *The Sea - Ocean Engineering Science, Part A*, Vol. 9, Wiley, New York, pp. 3–25.
Retyped with corrections: http://johndfenton.com/Papers/Fenton90b-Nonlinear-wave-theories.pdf.

Fenton, J. D. (1995), A numerical cnoidal theory for steady water waves, in *Proc. 12th Australasian Coastal and Ocean Engng Conference*, Melbourne, pp. 157–162.

Fenton, J. D. (1999), The cnoidal theory of water waves, in J. B. Herbich, ed., *Developments in Offshore Engineering*, Gulf, Houston, chapter 2, pp. 55–100.

Fenton, J. D. & Gardiner-Garden, R. S. (1982), Rapidly-convergent methods for evaluating elliptic integrals and theta and elliptic functions, *J. Austral. Math. Soc. Ser. B* **24**, 47–58.

Fenton, J. D. & McKee, W. D. (1990), On calculating the lengths of water waves, *Coastal Engineering* **14**, 499–513.

Hedges, T. S. (1995), Regions of validity of analytical wave theories, *Proc. Inst. Civ. Engnrs, Water, Maritime and Energy* **112**, 111–114.

Isobe, M., Nishimura, H. & Horikawa, K. (1982), Theoretical considerations on perturbation solutions for waves of permanent type, *Bull. Faculty of Engng, Yokohama National University* **31**, 29–57.

Le Méhauté, B., Divoky, D. & Lin, A. (1968), Shallow water waves: a comparison of theories and experiments, in *Proc. 11th Int. Conf. Coastal Engng*, Vol. 1, London, pp. 86–107.

Nelson, R. C. (1987), Design wave heights on very mild slopes - an experimental study, *Civ. Engng Trans, Inst. Engnrs Austral.* **CE29**, 157–161.

Nelson, R. C. (1994), Depth limited design wave heights in very flat regions, *Coastal Engineering* **23**, 43–59.

Rienecker, M. M. & Fenton, J. D. (1981), A Fourier approximation method for steady water waves, *J. Fluid Mechanics* **104**, 119–137.

Sobey, R. J., Goodwin, P., Thieke, R. J. & Westberg, R. J. (1987), Application of Stokes, cnoidal, and Fourier wave theories, *J. Waterway Port Coastal and Ocean Engng* **113**, 565–587.

Stokes, G. G. (1847), On the theory of oscillatory waves, *Transactions of the Cambridge Philosophical Society* **8**, 441–455. reprinted in: G.G. Stokes (1880) *Mathematical and Physical Papers*, Volume I. Cambridge University Press, pp. 197-229, http://www.archive.org/details/mathphyspapers01stokrich.

Williams, J. M. (1981), Limiting gravity waves in water of finite depth, *Phil. Trans. Roy. Soc. London Ser. A*

302(1466), 139–188.

Appendix A. Theory

Here we present an outline of the theory, initially for steady flow in (X, Y) co-ordinates. If the fluid is incompressible, in two dimensions a stream function $\psi(X, Y)$ exists such that the velocity components are given by

$$U = \partial\psi/\partial Y, \quad \text{and} \quad V = -\partial\psi/\partial X.$$

If motion is irrotational, then $\nabla \times \mathbf{u} = \mathbf{0}$ and it follows that ψ satisfies Laplace's equation throughout the fluid:

$$\frac{\partial^2\psi}{\partial X^2} + \frac{\partial^2\psi}{\partial Y^2} = 0. \tag{A-1}$$

The kinematic boundary conditions to be satisfied are

$$\psi(X, 0) = 0 \quad \text{on the bottom, and} \tag{A-2}$$
$$\psi(X, \eta(X)) = -Q \quad \text{on the free surface,} \tag{A-3}$$

where $Y = \eta(X)$ on the free surface and Q is a positive constant denoting the volume rate of flow per unit length normal to the flow underneath the stationary wave in the (X, Y) co-ordinates. In these co-ordinates the apparent flow is in the negative X direction. The dynamic boundary condition to be satisfied is that pressure is zero on the surface so that Bernoulli's equation becomes

$$\frac{1}{2}\left(\left(\frac{\partial\psi}{\partial X}\right)^2 + \left(\frac{\partial\psi}{\partial Y}\right)^2\right) + g\eta = R \quad \text{on the free surface,} \tag{A-4}$$

where R is a constant.

The basis of the method is to write the analytical solution for ψ in separated variables form

$$\psi(X, Y) = -\bar{U}\,Y + \sqrt{\frac{g}{k^3}}\sum_{j=1}^{N} B_j \frac{\sinh jkY}{\cosh jkd}\cos jkX, \tag{A-5}$$

where $\bar{U}$ is the mean fluid speed on any horizontal line underneath the stationary waves, the minus sign showing that in this frame the apparent dominant flow is in the negative x direction. The $B_{1}, \ldots, B_N$ are dimensionless constants for a particular wave, and N is a finite integer. The truncation of the series for finite N is the only mathematical or numerical approximation in this formulation. The quantity k is the wavenumber $k = 2\pi/\lambda$ where λ is the wavelength, which may or may not be known initially, and d is the mean depth as shown on Figure 4-1. Each term of this expression satisfies the field equation (A-1) and the bottom boundary condition (A-2) identically. The use of the denominator $\cosh jkd$ is such that for large j the B_j do not have to decay exponentially, thereby making solution rather more robust. For points on the free surface, where $Y = \eta$, for large jkd

$$\frac{\sinh jk\eta}{\cosh jkd} \sim e^{jk(\eta-d)},$$

not nearly as large as the numerator and denominator would be.

If one were proceeding to an analytical solution, the coefficients B_j would be found by using a perturbation expansion in wave height. Here they are found numerically by satisfying the two nonlinear equations (A-3) and (A-4) from the surface boundary conditions. Substituting $Y = d + \eta(X)$, into the surface equations and introducing the variables $q = \bar{U}d - Q$, the volume flux due to the waves which is actually a positive quantity, and $r = R - gd$, the energy per unit mass with datum at the mean water level, and dividing through to make the equations dimensionless:

$$\sum_{j=1}^{N} B_j \left[\frac{\sinh jk\eta}{\cosh jkd}\right]\cos jkX - \bar{U}\sqrt{k/g}\,k\,(\eta - d) - q\sqrt{\frac{k^3}{g}} = 0, \quad \text{and} \tag{A-6}$$

$$\frac{1}{2}\left(-\bar{U}\sqrt{k/g} + \sum_{j=1}^{N} jB_j\left[\frac{\cosh jk\eta}{\cosh jkd}\right]\cos jkX\right)^2 + \frac{1}{2}\left(\sum_{j=1}^{N} jB_j\left[\frac{\sinh jk\eta}{\cosh jkd}\right]\sin jkX\right)^2 + k\,(\eta - d) - rk/g = 0, \tag{A-7}$$

both to be satisfied for all x. In both equations, we will never evaluate the terms in square brackets as they are written, for both numerators and denominators can become very large. Instead we re-write them and evaluate them in the forms

$$C_j(kd, k\eta) = \frac{\cosh jk\eta}{\cosh jkd} = \cosh\left(jk\left(\eta - d\right)\right) + \tanh jkd \, \sinh\left(jk\left(\eta - d\right)\right), \quad \text{(A-8a)}$$

$$S_j(kd, k\eta) = \frac{\sinh jk\eta}{\cosh jkd} = \sinh\left(jk\left(\eta - d\right)\right) + \tanh jkd \, \cosh\left(jk\left(\eta - d\right)\right). \quad \text{(A-8b)}$$

The two hyperbolic functions of $jk\left(\eta - d\right)$ here are much smaller than $\cosh jk\eta$, while the $\tanh jkd$ function is not a problem, as it simply goes to 1 for large arguments.

To solve the problem numerically these two equations are to be satisfied at a sufficient number of discrete points so that we have enough equations for solution. If we evaluate the equations at $N+1$ discrete points over one half wave from the crest to the trough for $m = 0, 1, \ldots, N$, such that $X_m = m\lambda/2N$ and $kX_m = m\pi/N$, and where $\eta_m = \eta(X_m)$, then (A-6) and (A-7) provide $2N+2$ nonlinear equations in the $2N+5$ dimensionless variables: $k\eta_m$ for $m = 0, 1, \ldots, N$; B_j for $j = 1, 2, \ldots, N$; $\bar{U}\sqrt{k/g}$; kd; $q\sqrt{k^3/g}$; and rk/g. We now consider more equations and variables.

An extra equation is the expression requiring that the mean of the dimensionless depths $k\eta_m$ be kd, simply using the trapezoidal rule:

$$\frac{1}{N}\left(\frac{1}{2}\left(k\eta_0 + k\eta_N\right) + \sum_{m=1}^{N-1} k\eta_m\right) - kd = 0. \quad \text{(A-9)}$$

For quantities which are periodic such as here, the trapezoidal rule is very much more accurate than usually believed. It can be shown that the error is of the order of the last (Nth) coefficient of the Fourier series of the function being integrated. As that is essentially the approximation used throughout this work, where it is assumed that the series can be truncated at a finite value of N, this is in keeping with the overall accuracy.

In practice the physical dimensions of mean water depth d and wave height H are known giving a numerical value of H/d for which an equation can be provided connecting the crest and trough heights $k\eta_0$ and $k\eta_N$ respectively: $H = \eta_0 - \eta_N$,which we write in terms of our dimensionless variables as

$$k\eta_0 - k\eta_N - kd\,\frac{H}{d} = 0. \quad \text{(A-10)}$$

A.1 Specification of wavelength

In some problems we know the wavelength λ and so we have a numerical value for kd, that we write here as an equation

$$kd - 2\pi\frac{d}{\lambda} = 0, \quad \text{(A-11)}$$

and the problem is now closed.

A.2 Specification of wave period and current

In many problems it is not the wavelength λ which is known but the wave period τ as measured in a stationary frame. The two are connected by the simple relationship

$$c = \frac{\lambda}{\tau}, \quad \text{(A-12)}$$

where c is the wave speed, however it is not known *a priori*, and in fact depends on the current on which the waves are travelling. In the frame travelling with the waves at velocity c the mean horizontal fluid velocity at any level is $-\bar{U}$, hence in the stationary frame the time-mean horizontal fluid velocity at any point denoted by $\bar{u}_1$, the mean current which a stationary meter would measure, is given by

$$\bar{u}_1 = c - \bar{U}. \quad \text{(A-13)}$$

In the special case of no mean current at any point, $\bar{u}_1 = 0$ and $c = \bar{U}$, which is Stokes' first approximation to the wave speed, usually incorrectly referred to as his ”first definition of wave speed”, and is that relative to a frame in which the current is zero. Most wave theories have presented an expression for $\bar{U}$, obtained from its definition as a mean fluid speed. It has often been referred to, incorrectly, as ”the wave speed”.

A second type of mean fluid speed or current is the depth-integrated mean speed of the fluid under the waves in the frame in which motion is steady. If Q is the volume flow rate per unit span underneath the waves in the (X, Y) frame, the depth-averaged mean fluid velocity is $-Q/d$, where d is the mean depth. In the physical (x, y) frame, the depth-averaged mean fluid velocity, the ”mass-transport velocity”, is $\bar{u}_2$, given by

$$\bar{u}_2 = c - Q/d. \tag{A-14}$$

If there is no mass transport, such as in a closed wave tank, $\bar{u}_2 = 0$, and Stokes’ second approximation to the wave speed is obtained: $c = Q/d$. In general, neither of Stokes’ first or second approximations is the actual wave speed. Usually the overall physical problem will impose a certain value of current on the wave field, thus determining the wave speed. To apply the methods of this section, where wave period is known, to obtain a unique solution it is also necessary to specify the magnitude and nature of that current.

Appendix B. Program details

B.1 Initial solution

We calculate the initial values from linear wave theory, assuming zero current. The well-known solution for angular frequency $\sigma = 2\pi/\tau$ in terms of kd is

$$\sigma^2 = gk \tanh kd. \tag{B-1}$$

If the wave period and hence σ is known, it is necessary to solve for kd. The equation could be solved using standard methods for solution of a single nonlinear equation, however Fenton & McKee (1990) have given an approximate explicit solution:

$$kd \approx \frac{\sigma^2 d}{g}\left(\coth\left(\left(\sigma\sqrt{d/g}\right)^{3/2}\right)\right)^{2/3}. \tag{B-2}$$

This expression is an exact solution of (B-1) in the limits of long and short waves, and between those limits its greatest error is 1.5%. Such accuracy is adequate for the present approximate purposes. Having solved for kd, linear theory can be applied for an assumption of zero current in relating wavelength and period. We set $q = 0$, $r = \bar{U}^2/2$ (noting that it would have been nicer if Fenton (1988) had defined r as being solely due to the wave motion $r = R - gd - \bar{U}^2/2$, so that it would be zero in the small wave limit). We assume $\eta = d + \frac{1}{2}H \cos kX$ and substitute into the surface equations with B_1 non-zero and all higher terms in the series zero. The kinematic boundary condition (A-6) gives, taking the terms in the square brackets to be unity, thereby linearising, and dropping factors of $\cos kX$:

$$B_1 \tanh kd - \bar{U}\sqrt{k/g}\,\frac{kH}{2} = 0, \tag{B-3}$$

and repeating for the dynamic boundary condition (A-7), expanding the terms in brackets, considering only linear terms, and dropping factors of $\cos kX$:

$$-\bar{U}\sqrt{k/g}B_1 + \frac{kH}{2} = 0, \tag{B-4}$$

with solutions $\bar{U}\sqrt{k/g} = \sqrt{\tanh kd}$, that we could have written down from equation (B-1), making the zero current approximation $\bar{U} = c$. What is more useful here is the other solution obtained from the pair of equations, $B_1 = kH/2/\sqrt{\tanh kd}$. Hence we have the linear solution, and in view of the common occurrence, we use the symbol $C_0 = \sqrt{\tanh kd}$ borrowed from Fenton (1985).

$$\begin{aligned}
k\eta_m &= kd + \frac{1}{2}kH\cos\frac{m\pi}{N}, \quad \text{for } m = 0,\ldots,N, \\
\bar{U}\sqrt{k/g} &= c\sqrt{k/g} = C_0, \\
B_1 &= \frac{1}{2}\frac{kH}{C_0}, \quad B_j = 0 \text{ for } j = 2,\ldots,N, \\
q &= 0, \\
rk/g &= \tfrac{1}{2}\frac{\bar{U}^2 k}{g} = \tfrac{1}{2}C_0^2.
\end{aligned}$$

For the currents, we use $\bar{u}_1$ or $\bar{u}_2$, if we have a value. Otherwise we assume zero.

B.2 Dimensionless variables

Here we set out and number all the variables above, made dimensionless with respect to ρ, g, and wavenumber k, and in the last column add the initial linear solution. Once an initial value of kd has been calculated, all other quantities can be calculated sequentially.

Variable reference number j	Dimensionless variable z_j	Physical quantity	Initial value from linear theory
1	kd	Depth	Known kd or eqn (B-2)
2	kH	Wave height	$kd \times (H/d)$
3	$\tau\sqrt{gk}$	Period	$2\pi/C_0$
4	$c\sqrt{k/g}$	Wave speed	C_0
5	$\bar{u}_1\sqrt{k/g}$	Mean Eulerian current	0 or $\sqrt{kH} \times \bar{u}_1/\sqrt{gH}$
6	$\bar{u}_2\sqrt{k/g}$	Mean Stokes current	$\sqrt{kH} \times \bar{u}_2/\sqrt{gH}$ or 0
7	$\bar{U}\sqrt{k/g}$	Mean fluid speed in frame of wave	C_0
8	$q\sqrt{k^3/g}$	Discharge due to waves	0
9	rk/g	Energy due to waves	$\frac{1}{2}C_0^2$
$10,\ldots,N+10$	$k\eta_m$, $m = 0..N$	$N+1$ surface elevations	$kd + \frac{1}{2}kH\cos\frac{m\pi}{N}$
$N+11,\ldots,2N+10$	B_m, $m = 1..N$	N Fourier coefficients	$B_1 = \frac{1}{2}kH/C_0$, $B_2 = 0, \ldots$

Table B-1. List of dimensionless variables to be determined

B.3 Equations

In all the following it is assumed that values of H and d are known, plus a value of either λ or values of τ and either $\bar{u}_1$ or $\bar{u}_2$.

Equation 1 – Wave height in terms of H/d

$$kH - kd \times (H/d) = 0$$

Equation 2 – Wave height in terms of wavelength or period, whichever is known

$$\begin{aligned}
&kH - (H/\lambda) \times 2\pi = 0 \quad \text{or} \\
&kH - \left(H/g\tau^2\right) \times \left(\tau\sqrt{gk}\right)^2 = 0.
\end{aligned}$$

Equation 3 – Definition of wave speed $c = \lambda/\tau$, equation (A-12) in dimensionless terms,

$$c\sqrt{k/g} \times \tau\sqrt{gk} - 2\pi = 0.$$

Equation 4 – Mean Eulerian current, equation (A-13)

$$\bar{u}_1\sqrt{k/g}+\bar{U}\sqrt{k/g}-c\sqrt{k/g}=0.$$

Equation 5 – Mean mass-transport current, equation (A-14) converted to use q

$$\bar{u}_2\sqrt{k/g}+\bar{U}\sqrt{k/g}-c\sqrt{k/g}-\frac{q\sqrt{k^3/g}}{kd}=0.$$

Equation 6 – From known or assumed numerical value of one current or the other $\bar{u}_1/\sqrt{gd}$ or $\bar{u}_2/\sqrt{gd}$ (May 2013: the program had been using $\sqrt{kH}$ rather than $\sqrt{kd}$ here)

$$\bar{u}_m\sqrt{k/g}-\frac{\bar{u}_m}{\sqrt{gd}}\times\sqrt{kd}=0,\quad \text{for } m=1 \text{ or } 2.$$

Equation 7 – Mean value of $k\eta$ is kd, equation (A-9)

$$\frac{1}{N}\left(\frac{1}{2}\left(k\eta_0+k\eta_N\right)+\sum_{m=1}^{N-1}k\eta_m\right)-kd=0.$$

Equation 8 – Definition of wave height, equation (A-10)

$$k\eta_0-k\eta_N-kH=0.$$

Equations 9 to $N+9$ – Kinematic free surface boundary condition (A-6): using equation (A-8b) and with $kX_m=m\pi/N$ for $m=0..N$:

$$\sum_{j=1}^{N}B_jS_j(kd,k\eta_m)\cos\frac{jm\pi}{N}-\bar{U}\sqrt{k/g}\,k\left(\eta_m-d\right)-q\sqrt{\frac{k^3}{g}}=0.$$

Equations $N+10$ to $2N+10$ – Dynamic free surface boundary condition (A-7): using equations (A-8) and with $kX_m=m\pi/N$ for $m=0..N$:

$$\frac{1}{2}\left(-\bar{U}\sqrt{k/g}+\sum_{j=1}^{N}jB_jC_j(kd,k\eta_m)\cos\frac{jm\pi}{N}\right)^2+\frac{1}{2}\left(\sum_{j=1}^{N}jB_jS_j(kd,k\eta_m)\sin\frac{jm\pi}{N}\right)^2$$
$$+k\left(\eta_m-d\right)-rk/g=0.$$

B.4 Enumeration of variables and equations

From Table B-1 it can be seen that there are $2N+10$ variables and here we have written out $2N+10$ equations. Some formulations of the problem (*e.g.* Dean, 1965) allow more surface collocation points and the equations are solved in a least-squares sense. This is a good idea and in general would be thought to be desirable, but in practice seems not to make much difference, and here the procedure of Rienecker & Fenton (1981) and Fenton (1988) is followed, where the same numbers of equations as unknowns is used. In the computer program the numbering of variables follows that of Fenton (1988).

B.5 Computational method

The system of nonlinear equations can be iteratively solved using Newton's method. If we write the system of equations as

$$F_i(\mathbf{x})=0,\quad \text{for } i=1,\ldots,2N+10,$$

where F_i represents equation i and $\mathbf{x}=\{x_j,j=1,\ldots,2N+10\}$, the vector of variables x_j (there should be no confusion with that same symbol as a space variable), then if we have an approximate solution $\mathbf{x}^{(n)}$ after n iterations, writing a multi-dimensional Taylor expansion for the left side of equation i obtained by varying each of the $x_j^{(n)}$ by some increment $\delta x_j^{(n)}$:

$$F_i\left(\mathbf{x}^{(n+1)}\right) \approx F_i\left(\mathbf{x}^{(n)}\right) + \sum_{j=1}^{2N+10} \left(\frac{\partial F_i}{\partial x_j}\right)^{(n)} \delta x_j^{(n)}.$$

If we choose the $\delta x_j^{(n)}$ such that the equations would be satisfied by this procedure such that $F_i\left(\mathbf{x}^{(n+1)}\right) = 0$, then we have the set of linear equations for the $\delta x_j^{(n)}$:

$$\sum_{j=1}^{2N+10} \left(\frac{\partial F_i}{\partial x_j}\right)^{(n)} \delta x_j^{(n)} = -F_i\left(\mathbf{x}^{(n)}\right) \quad \text{for } i = 1, \ldots, 2N+10,$$

which is a set of equations linear in the unknowns $\delta x_j^{(n)}$ and can be solved by standard methods for systems of linear equations. Having solved for the increments, the updated values of all the variables are then computed for $x_j^{(n+1)} = x_j^{(n)} + \delta x_j^{(n)}$ for all the j. As the original system is nonlinear, this will in general not yet be the required solution and the procedure is repeated until it is.

It is possible to obtain the array of derivatives of every equation with respect to every variable, $\partial F_i/\partial x_j$ by performing the analytical differentiations, however as done in Fenton (1988), it is rather simpler to obtain them numerically. That is, if variable x_j is changed by an amount ε_j, then on numerical evaluation of equation i before and after the increment (after which it is reset to its initial value), we have the numerical derivative

$$\frac{\partial F_i}{\partial x_j} \approx \frac{F(x_1, \ldots, x_j + \varepsilon_j, \ldots, x_{2N+10}) - F(x_1, \ldots, x_j, \ldots, x_{2N+10})}{\varepsilon_j}.$$

The complete array is found by repeating this for each of the $2N+10$ equations for each of the $2N+10$ variables. Compared with the solution procedure, which is $O(N^3)$, this is not a problem, and gives a rather simpler program.

B.6 Post-processing to obtain quantities for practical use

Once the solution has been obtained, quantities rather more useful for physical calculations can be evaluated, notably surface elevations and velocities and accelerations.

It can be shown from (A-5) and the Cauchy-Riemann equations

$$\frac{\partial \Phi}{\partial X} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \Phi}{\partial y} = -\frac{\partial \psi}{\partial X},$$

where Φ is the velocity potential in the frame moving with the wave, and $X = x - ct$, such that

$$\Phi(X, y) = -\bar{U}\, X + \sqrt{\frac{g}{k^3}} \sum_{j=1}^{N} B_j \frac{\cosh jky}{\cosh jkd} \sin jkX.$$

In the physical frame, the now unsteady velocity potential $\phi(x, y, t)$ is written

$$\phi(x, y, t) = \Phi(x - ct, y) + c\,(x - ct)$$

such that the horizontal velocities in the two systems are related by

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \Phi}{\partial x} + c = U + c,$$

which result would have been obtained by just adding cx to Φ, but it is slightly simpler to have ϕ expressed also as a function of $x - ct$. The additional function of time does not affect the dynamics, it will merely affect the manner in which we subsequently write the unsteady Bernoulli equation. Now we have

$$\phi(x, y, t) = \left(c - \bar{U}\right)(x - ct) + \sqrt{\frac{g}{k^3}} \sum_{j=1}^{N} B_j \frac{\cosh jky}{\cosh jkd} \sin jk\,(x - ct)\,. \tag{B-5}$$

The velocity components anywhere in the fluid are given by $u = \partial\phi/\partial x$, $v = \partial\phi/\partial y$:

$$u = c - \bar{U} + \sqrt{\frac{g}{k}} \sum_{j=1}^{N} jB_j \frac{\cosh jky}{\cosh jkd} \cos jk(x-ct), \tag{B-6}$$

$$v = \sqrt{\frac{g}{k}} \sum_{j=1}^{N} jB_j \frac{\sinh jky}{\cosh jkd} \sin jk(x-ct), \tag{B-7}$$

and as ϕ is a function of $x - ct$ we have simply

$$\frac{\partial\phi}{\partial t} = -c\frac{\partial\phi}{\partial x} = -cu. \tag{B-8}$$

Acceleration components can be obtained simply from these expressions by differentiation, and from the Cauchy-Riemann equations, and are given by.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -c \times \frac{\partial u}{\partial x}, \quad \text{where} \quad \frac{\partial u}{\partial x} = -\sqrt{gk}\sum_{j=1}^{N} j^2 B_j \frac{\cosh jky}{\cosh jkd} \sin jk(x-ct), \\
\frac{\partial v}{\partial t} &= -c \times \frac{\partial v}{\partial x}, \quad \text{where} \quad \frac{\partial v}{\partial x} = \sqrt{gk}\sum_{j=1}^{N} j^2 B_j \frac{\sinh jky}{\cosh jkd} \cos jk(x-ct), \\
\frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x}, \\
\frac{\partial v}{\partial y} &= -\frac{\partial u}{\partial x}.
\end{aligned}$$

The total material accelerations of a fluid particle are then

$$\begin{aligned}
\frac{Du}{Dt} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}, \quad \text{and} \\
\frac{Dv}{Dt} &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}.
\end{aligned}$$

The free surface elevation at an arbitrary point requires another step, as we only have it at discrete points $k\eta_m$, obtained as part of the solution. We take the cosine transform of the $N+1$ surface elevations:

$$E_j = \frac{2}{N} \sum_{m=0}^{N}{}'' k\eta_m \cos\frac{jm\pi}{N} \quad \text{for } j = 0, \ldots, N,$$

where Σ'' means that it is a trapezoidal-type summation, with factors of 1/2 multiplying the first and last contributions. The cosine transform could be performed using fast Fourier methods, but as N is not large, simple evaluation of the series is reasonable. It is easily shown that $E_0 = 2kd$, twice the mean value of the $k\eta_m$. The interpolating cosine series for the surface elevation is then

$$k\eta(x,t) = \sum_{j=0}^{N}{}'' E_j \cos jk(x-ct), \tag{B-9}$$

which can be evaluated for any x and t, taking care to use the trapezoidal sum. Of course the first term is $\frac{1}{2} \times E_0 = kd$, the dimensionless mean depth.

The pressure at any point in the fluid can be evaluated using Bernoulli's theorem, but most simply in the form from the steady flow, but using the velocities as computed from (B-6) and (B-7):

$$\frac{p}{\rho} = R - gy - \frac{1}{2}\left((u-c)^2 + v^2\right). \tag{B-10}$$

We consider how this relates to the unsteady Bernoulli equation

$$\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + gy + \frac{1}{2}\left(u^2 + v^2\right) = f(t), \tag{B-11}$$

where $f(t)$ is an arbitrary function of time, determined by boundary conditions. Substituting equation (B-10) for pressure into this gives

$$\frac{\partial \phi}{\partial t} + R - f(t) + cu - \frac{1}{2}c^2 = 0.$$

From equation advect we have $\partial\phi/\partial t + cu = 0$, giving the expression for f which is, in fact, a constant:

$$f = R - \frac{1}{2}c^2,$$

so we have the unsteady Bernoulli equation in the form

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gy + \frac{1}{2}\left(u^2 + v^2\right) - \left(R - \frac{1}{2}c^2\right) = 0. \tag{B-12}$$

As a partial check on the subroutine POINT included in the C++ program, where $\partial\phi/\partial t$, velocities u and v (and accelerations) are calculated, it also calculates the value of the left side of this equation.

Appendix C. Theoretical and computational aspects of CNOIDAL

C.1 Summary

Fenton (1979) presented a method for the computer generation of high-order cnoidal solutions for periodic waves. Results were expressed in terms of the relative wave height to depth. It was shown how it is rather simpler to use the trough depth h as the depth scale in presenting results, and that the effective expansion parameter is not simply H/h but is actually that divided by the elliptic parameter m used in the theory. For the expansion parameter to be small and for the series results to be valid, the short wave limit is excluded. In this way the cnoidal theory breaks down in deep water (short waves) in a manner complementary to that in which Stokes theory breaks down in shallow water (long waves), (Fenton 1985). For moderate waves, results were good when compared with experiment, but for higher waves the velocity profile showed exaggerated oscillations and it was found that ninth-order results were worse. These results were unexpectedly poor.

In a review article the author (Fenton 1990) considered cnoidal theory as well as Stokes theory and Fourier approximation methods. The approach to cnoidal theory in Fenton (1979) was re-examined. It was found that if the series for velocity were expressed in terms of the shallowness (effectively $(\text{depth}/\text{length})^2$) rather than relative wave height, as was done by Chappelear (1962), then results were very much better, and justified the use of cnoidal theory even for high waves. This is in accordance with the fundamental approximation of the cnoidal theory being an expansion in shallowness. That review article also incorporated the fact that the wave theory does not determine the wave speed, and that neither Stokes' first nor second approximations to wave speed are necessarily correct. In general it is necessary to incorporate the effects of current, as had been done for numerical wave theories in Rienecker & Fenton (1981) and for high-order Stokes theory in Fenton (1985).

A rather longer review article of cnoidal theory was presented by Fenton (1999).

C.2 Alternative expressions for elliptic quantities

As m is close to unity for most applications of the cnoidal theory, a huge problem is presented for many practical problems, and that is because the formulae presented in references such as Abramowitz & Stegun (1965)[1] are most rapidly convergent in the limit $m \to 0$, and may be slowly convergent, if at all, in the other limit, which is of interest to us here. Such methods as the Landen transform are presented, for which convergence is assured, but that is a procedure, rather than explicit approximations. Fenton & Gardiner-Garden (1982) used Jacobi's imaginary transformation to recast the better-known expressions for theta functions so that they are most rapidly convergent in the limit as the parameter tends to unity. These results were then used to give alternative expressions for the elliptic functions which also converge most rapidly in the limit where previously-presented expressions do not converge. Finally, alternative methods for the calculation of complete elliptic integrals were developed. These are shown to

[1] http://people.math.sfu.ca/~cbm/aands/frameindex.htm

be the simple complement of well-known methods but, remarkably, seem to be unknown. These expressions have been implemented in the accompanying program.

C.3 Series enhancement

Fenton (1972) used what, at the time of the 1970s, were known as Shanks transforms to transform the series and improve convergence of solitary wave solutions. In the meantime, it has been discovered that the original technique goes back to Aitken, and they are now referred to as Aitken transformations. They take a series and attempt to mimic the behaviour of the series as if it had an infinite number of terms. The method takes three successive terms in a sequence (such as the first, second and third order solutions for a wave property), and extrapolates the behaviour of the sequence to infinity, hopefully mimicking the behaviour of the series if there were an infinite number of terms. The method can work surprisingly well. It is easily implemented: if the last three terms in a sequence of n terms are S_{n-2}, S_{n-1}, and S_n, an estimate of the value of S_∞, denoted by S'_n is given by

$$S'_n \approx S_n - \frac{(S_n - S_{n-1})^2}{(S_n - S_{n-1}) - (S_{n-1} - S_{n-2})}. \tag{C-1}$$

This is the form which is most suitable for computations, when in the possible case that the sums have nearly converged and both numerator and denominator of the second term on the right go to zero the result is less liable to round-off error.

The transform is simply applied and can be used in many areas of numerical computations. It gives surprisingly good results, but its theoretical justification is limited and sometimes it does not work well. Whereas Padé approximants are more general, Aitken transforms are rather simpler to apply – one just has to have three numbers to obtain a better estimate of their likely converged result. The author has tested the use of both Aitken transforms and Padé approximants in applying the cnoidal theory described in this work. A limitation to Padé approximation became quickly obvious, when at the first step in practical application, solving an equation for the wavelength, approximating a quartic in H/d by a $[2/2]$ Padé approximant, with a quadratic in numerator and denominator, the latter passed through zero for an intermediate value of H/d, such that in the vicinity of that point very wildly varying results were obtained. The author considered that this was sufficiently dangerous that generally Padé approximants could not be recommended for the approximation of fifth-order cnoidal theory.

In practice, obtaining solutions for given values of wavelength and wave height, the use of the Aitken transforms everywhere gave better results than just using the raw series in the case of global wave quantities such as α, Q, *etc.* which are independent of position, and it is recommended that Aitken transforms be used to improve the accuracy of all series computations for those quantities. They are of course trivially implemented, given say, three numbers.

However, for fluid velocities it has been found that Aitken transforms could lead to irregularities in the results, and the program uses un-transformed expressions for the velocity fields, with results such as have been presented in Figures 8-1 and 8-2 above.

C.4 Initial solution for the elliptic integrals

The quantity m is a parameter that occurs throughout all the cnoidal theory. For long waves, for which cnoidal theory is applicable, it is usually close to 1 and indeed is usually remarkably close to unity, which is why the alternative expressions for elliptic quantities described above were developed. For a solitary wave, of infinite wavelength, it is $m = 1$. A computational problem is that it has to be calculated from the wave data before anything else, which involves solving a transcendental equation, in which part of the equation varies strongly logarithmically with m as $m \to 1$. In Fenton (1999, p14) it was shown that, at first order a simple approximation for m in terms of Ursell number Ur and the Stokes-Ursell number SU from equation (3.3) gives alternative expressions:

$$m \approx 1 - 16\, e^{-\sqrt{3\mathrm{Ur}/4}} = 1 - 16\, e^{-(\lambda/d)\sqrt{3H/4d}} \approx 1 - e^{3.-8.\sqrt{\mathrm{SU}}}. \tag{C-2}$$

In view of Hedges' suggested limitation (Hedges 1995) that cnoidal theory be applied only for $\mathrm{Ur} > 40$ (or $\mathrm{SU} > 1/2$) it follows that m is always greater than 0.93 if the theory is used within its recommended limit. In many practical problems, the waves are so long that m is very close to unity. For example, with a value of $\mathrm{SU} = 1$, just twice the recommended minimum value, $m \approx 0.99$.

Now there are different equations to solve to improve this estimate, presuming that the initially-known quantities are dimensionless wave height H/d and either dimensionless wavelength λ/d or period $\tau\sqrt{g/d}$ plus dimensionless current $\bar{u}_1/\sqrt{gd}$ or $\bar{u}_2/\sqrt{gd}$. Consider Fenton (1999, eqn A.7) for wavelength in terms of the known H/d plus m and $e(m) = E(m)/K(m)$, which is the ratio of elliptic integrals of the first and second kind, giving to second order

$$\frac{\lambda}{d} = 4K(m)\left(3\frac{H}{md}\right)^{-1/2}\left(1+\left(\frac{H}{md}\right)\left(\frac{5}{4}-\frac{5}{8}m-\frac{3}{2}e(m)\right)+\left(\frac{H}{md}\right)^2(\ldots)+\ldots\right). \quad \text{(C-3)}$$

This is a transcendental equation for m, given numerical values of H/d and λ/d. The variation of $K(m)$ with m is very rapid in the limit as $m \to 1$, as it contains a singularity in that limit, hence, gradient methods such as the secant method for this might break down. The author originally preferred to use a bisection method, for which reference can be made to any introductory book on numerical methods. However, in preparing the current computer program and this note, the following simpler approach was suggested, re-writing equation (C-3) in terms of $K(m)$:

$$K(m) = \frac{1}{4}\frac{\lambda}{d}\frac{\left(3\frac{H}{md}\right)^{1/2}}{1+\left(\frac{H}{md}\right)\left(\frac{5}{4}-\frac{5}{8}m-\frac{3}{2}\frac{E(m)}{K(m)}\right)+\ldots}. \quad \text{(C-4)}$$

The variation of the right side with $K(m)$ is not very strong, and so this expression can be evaluated in an iterative sense, at each iteration using the procedure suggested at the end of Fenton & Gardiner-Garden (1982) to obtain m from the iterated value of K. This procedure worked very well, requiring little programming and with convergence rapid and assured.

In the case of wave period being known, the two alternative approaches are:

Eulerian current $\bar{u}_1/\sqrt{gd}$ known: equation (A-13) can be re-arranged to give

$$K(m) = \frac{1}{4}\tau\sqrt{\frac{g}{d}}\left(\frac{\bar{u}_1}{\sqrt{gd}}+\frac{\bar{U}}{\sqrt{gh}}\sqrt{\frac{h}{d}}\right)\frac{\left(3\frac{H}{md}\right)^{1/2}}{1+\left(\frac{H}{md}\right)\left(\frac{5}{4}-\frac{5}{8}m-\frac{3}{2}\frac{E(m)}{K(m)}\right)+\ldots}, \quad \text{(C-5)}$$

where $\bar{U}/\sqrt{gh}$ is given as a function of $\varepsilon = H/h = (H/d)\,/\,(h/d)$ and m and $e(m)$ in the theory, and where in the equation and in ε, h/d is given as a series in H/d (known) and m and $e(m)$. To write out the expressions would be very complicated; to evaluate them numerically by computer is not. Equation (C-5) can be solved iteratively in the same manner as for equation (C-4).

Stokes current $\bar{u}_2/\sqrt{gd}$ known: equation (A-14) can be re-arranged in the same manner to give

$$K(m) = \frac{1}{4}\tau\sqrt{\frac{g}{d}}\left(\frac{\bar{u}_2}{\sqrt{gd}}+\frac{Q}{\sqrt{gh^3}}\left(\frac{h}{d}\right)^{3/2}\right)\frac{\left(3\frac{H}{md}\right)^{1/2}}{1+\left(\frac{H}{md}\right)\left(\frac{5}{4}-\frac{5}{8}m-\frac{3}{2}\frac{E(m)}{K(m)}\right)+\ldots}, \quad \text{(C-6)}$$

where the dimensionless discharge $Q/\sqrt{gh^3}$ is given as a function of $\varepsilon = H/h = (H/d)\,/\,(h/d)$ and m and $e(m)$ in the theory.

C.5 Modification in July 2019

Thomas Lykke Andersen of the Department of Civil Engineering at Aalborg University wrote to me saying that for very long and high waves, when Ur is large, as $m \approx 1 - 16\,e^{-\sqrt{3\mathrm{Ur}/4}}$ (equation C-2), the value of m is so close to 1 that when when the elliptic integral of the first kind $K(m)$ was evaluated using the approximations of Fenton & Gardiner-Garden (1982), requiring the evaluation of $\log(1-m)$, the round-off error in evaluating $1-m$ gave it as zero, for which, of course, the logarithmic function failed. The program has been modified for problems in which $m_1 = 1 - m \approx 16\,e^{-\sqrt{3\mathrm{Ur}/4}}$ is smaller than, say 10^{-8} such that m can only be stored correct to some six figures. In such cases, initially it no longer solves for m, but instead for the value of K, as appearing in the expressions in the previous section, while everywhere else setting $m = 1$.

Also, in evaluating all the elliptic functions and integrals, instead of using the general subroutines, the simpler approximations of Fenton & Gardiner-Garden (1982), as shown in Table 2 of Fenton (1999) are now used, so that the programs are simpler. The approximations are accurate to 5 figures for $m > \frac{1}{2}$, so for the values of $m > 0.9$ used for cnoidal theory, they are very accurate.